Extraordinary Solution in the Flood–Garber Model of Financial Crises

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ABSTRACT

The Flood–Garber equation in a classical model of financial crises has an extraordinary solution which gives an exponential increase of the shadow exchange rate $e_s(t)$. In the Flood and Garber model, the shadow exchange rate $e_s(t)$ is assumed as a linear function in t, so that the collapse time t_c can be predicted by using observed quantities at $t < t_c$. However, such the assumption cannot be justified at all, so that in general we cannot predict the collapse time t_c. In the present paper, it is pointed out that this problem originates in a more fundamental property of the differential equation: a solution of an n-th order differential equation inevitably contains n integration constants, which are fixed only by boundary conditions (initial conditions); since the behavior of $e_s(t)$ is unknown for $t < t_c$, we cannot use boundary conditions at $t < t_c$, but we can use only those at $t \geq t_c$; this means that the boundary condition at $t = t_c$ is incompetent to predict the collapse time t_c.

1. Introduction

A model of financial crises proposed by Flood and Garber [1] is nowadays called as a classical model[2] of financial crises. However, the model is still useful for understanding the basic concept and fundamental framework of financial crises[3].

The purpose of the present paper is not to propose a practically useful mathematical model for the financial crises, but to demonstrate how important a careful study of differential equations is in a mathematical model. From a careful study of the differential equation which has been proposed by Flood and Garber, we will point out that we cannot predict the collapse time t_c even if the domestic credit function $D(t)$ is linear as assumed by Flood and Garber, and this problem originates in a more general property of the differential equation.

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[2] Classical models have been proposed by Krugman [2] and proceeded by Flood and Garber [1]. In contrast to these classical models, there are recent models which are based on a concept "self-fulling" [3].

[3] For a review, for example, see Ref. [4].

The Flood–Garber equation is given by

$$M(t) = \beta e(t) - \alpha \frac{de(t)}{dt} \ . \tag{1.1}$$

Here, $M(t)$ and $e(t)$ are a money supply and a domestic exchange rate, respectively. The coefficients α and β are defined as follows. We assume that (money supply $M(t)$/domestic price level $P(t)$) is approximated as a linear form in a domestic interest rate $i(t)$:

$$\frac{M(t)}{P(t)} = a_0 - a_1 i(t) \ , \tag{1.2}$$

where $a_0 > 0$ and $a_1 > 0$. The domestic exchange rate $e(t)$ is related to the domestic price level $P(t)$ and the foreign price level $P_w(t)$ as

$$P(t) = P_w(t) e(t) \ , \tag{1.3}$$

and the domestic interest rate $i(t)$ is related to the foreign interest rate $i_w(t)$ as

$$i(t) = i_w(t) + \frac{1}{e(t)} \frac{de(t)}{dt} \ . \tag{1.4}$$

By substituting Eqs. (1.3) and (1.4) into Eq. (1.2), we obtain the differential equation

$$M = \left[a_0 - a_1 \left(i_w + \frac{1}{e} \frac{de}{dt} \right) \right] P = (a_0 - a_1 i_w) P_w e - a_1 P_w \frac{de}{dt} \ , \tag{1.5}$$

so that by putting

$$\alpha = a_1 P_w \quad , \quad \beta = (a_0 - a_1 i_w) P_w \ , \tag{1.6}$$

we obtain the differential equation (1.1). Here, we have considered that the foreign price level $P_w(t)$ is almost constant (we take an interest only in the behavior of the domestic price level $P(t)$).

Generally, the money supply $M(t)$ is given by the sum of the rest of foreign reserves (the domestic foreign money holdings) $R(t)$ and the domestic credit $D(t)$,

$$M(t) = R(t) + D(t) \ . \tag{1.7}$$

In the fixed exchange rate regime, since the exchange rate $e(t)$ has to be always kept as a constant value $\bar{e}$, i.e.

$$\frac{de(t)}{dt} = \frac{d\bar{e}}{dt} = 0 \ , \tag{1.8}$$

we obtain, from Eq. (1.1),

$$M(t) = \beta e(t) = \beta \bar{e} \ . \tag{1.9}$$

Therefore, from (1.7), we see that the increase of the domestic credit function $D(t)$ inevitably causes the decrease of the international reserves $R(t)$. When $R(t) = 0$, the government is forced to abandon the fixed exchange rate regime for the floating exchange rate regime.

A concept of "shadow floating exchange rate" has been introduced by Flood and Garber [1]. The shadow floating exchange rate $e_s(t)$ is defined as an exchange rate which would virtually materialize if the government, which adopts the fixed exchange rate regime, adopted the floating exchange rate regime.

In Fig. 1, we illustrate behavior of the shadow floating exchange rate $e_s(t)$, where $e_s(t)$ increases in the time t and $e_s(t)$ exceeds the fixed exchange rate $e(t) = \bar{e}$ at $t = t_c$. For example, let us consider that someone speculates in money at $t = t_a$ ($t_a > t_c$), i.e. the speculator exchanges his huge domestic money for the foreign money with the exchange rate $\bar{e}$. Then, the government has to support the fixed exchange rate regime through intervention in currency market with foreign currency reserves. As a result, the rest of foreign reserves $R(t)$ is rapidly exhausted, and the fixed exchange rate regime collapses. Therefore, by selling the foreign money, the speculator can obtain the profit $e_a(t_a) - \bar{e}$. Such an attack has to start as soon as possible at $t > t_c$, because the later attacker cannot obtain the profit. In conclusion, the attack starts at $t = t_c + \varepsilon$ (ε is an infinitesimal positive value).

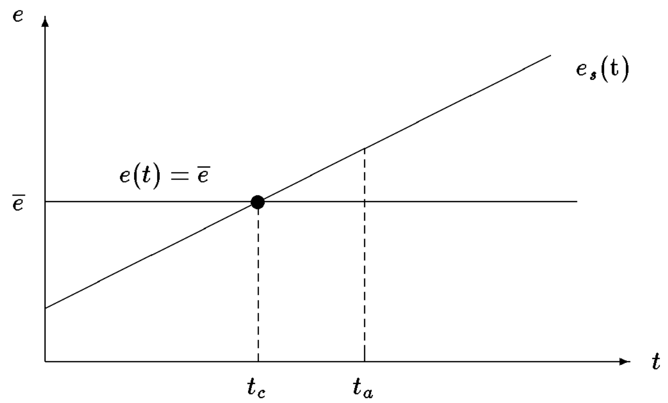

Fig. 1. Fixed exchange rate $\bar{e}$ and shadow floating exchange rate $e_s(t)$

In order to predict the collapse time $t = t_c$, we have to obtain the shadow floating exchange rate function $e_s(t)$. Since $e(t) = e_s(t)$ and $R(t) = 0$ for $t > t_c$, the function $e_s(t)$ is obtained by solving

$$D(t) = \beta e_s(t) - \alpha \frac{de_s(t)}{dt} \ , \tag{1.10}$$

However, this equation (1.10) cannot be solved without some subsidiary conditions. Flood and Garber have assumed a linear relation between the domestic credit $D(t)$ and the exchange rate $e(t)$,

$$e(t) = \lambda_0 + \lambda_1 D(t) \ , \tag{1.11}$$

in addition to a linear relation

$$\frac{dD(t)}{dt} = \mu > 0 \ , \tag{1.12}$$

and they have obtained

$$\lambda_1 = \frac{1}{\beta} \ , \quad \lambda_2 = \frac{\alpha\mu}{\beta^2} \ . \tag{1.13}$$

The results (1.13) are obtained as follows: by substituting (1.11) into de/dt in the differential equation (1.10), we obtain

$$\frac{dD(t)}{dt} = \frac{\beta\lambda_1 - 1}{\alpha\lambda_1} D(t) + \frac{\beta\lambda_0}{\alpha\lambda_1} \ . \tag{1.14}$$

From the requirement (1.12), the coefficient of $D(t)$ in the left-hand side of (1.14) must be zero, i.e. it must be $\beta\lambda_1 = 1$, and the factor $\beta\lambda_0/\alpha\lambda_1$ must be equal to μ, so that we can obtain the results (1.13). In conclusion, they have obtained the relation

$$e(t) = \frac{\mu}{\beta}(t - t_c) + \bar{e} \ , \tag{1.15}$$

and

$$D(t) = \mu(t - t_c) - \frac{\alpha}{\beta}\mu + \bar{e} \ . \tag{1.16}$$

On the other hand, from the relation (1.7), we obtain

$$\beta\bar{e} = R_0 + D_0 \ , \tag{1.17}$$

where $R_0 = R(0)$ and $D_0 = D(0)$ are the initial values of $R(t)$ and $D(t)$, respectively. By eliminating D_0 from Eq. (1.16) at $t = 0$ and Eq. (1.17), we can obtain

$$t_c = \frac{1}{\mu} R_0 - \frac{\alpha}{\beta} \ . \tag{1.18}$$

Since the quantities μ, α and β are known-values prior to the collapse, we can predict the collapse time $t = t_c$.

However, as we discuss in the next section, we will demonstrate that the differential equation (1.10) has an extraordinary solution, so-called A-term, which exponentially increases in time t, even under the assumption of the linear relation of $D(t)$, (1.12). Therefore, we will point out that we cannot predict the collapse time t_c from the known values at $t < t_c$. In Sec.3, we will show that this appearance of such an A-term is inevitable and independent of the structure of D(t). It will be pointed out that this problem originates in a more fundamental property of the differential equation.

2. Extraordinary solution

In the present section, we will show that the differential equation (1.10) has an extraordinary solution, i.e. non-linear solution, where the shadow exchange rate function $e_s(t)$ exponentially increases in t, even if we assume that the domestic credit function $D(t)$ is linear as given in (1.12).

In order to solve the differential equation (1.10) under the subsidiary condition (1.12), we differentiate Eq. (1.10) in t as

$$\frac{dD}{dt} = \beta\frac{de_s}{dt} - \alpha\frac{d^2e_s}{dt^2} \ , \tag{2.1}$$

so that we obtain the second order differential equation for $e_s(t)$,

$$\frac{d^2e_s}{dt^2} - \frac{\beta}{\alpha}\frac{de_s}{dt} + \frac{\mu}{\alpha} = 0 \ , \tag{2.2}$$

where we have used the subsidiary condition (1.12). Note that this equation (2.2) is not a linear differential equation. By putting

$$x = \frac{de_s}{dt} \ , \tag{2.3}$$

we obtain the equation

$$\frac{dx}{dt} - \frac{\beta}{\alpha}x + \frac{\mu}{\alpha} = 0 \ . \tag{2.4}$$

Then, by using the method of separation of variables, we obtain

$$\frac{dx}{x - \frac{\mu}{\beta}} = \frac{\beta}{\alpha}dt \ . \tag{2.5}$$

so that we obtain the solution

$$\log\left(x - \frac{\mu}{\beta}\right) = \frac{\beta}{\alpha}t + C_1 \ , \tag{2.6}$$

i.e.

$$x = Ae^{\frac{\beta}{\alpha}t} + \frac{\mu}{\beta} \ , \tag{2.7}$$

where we have put $e^{C_1} = A$. Therefore, recalling the relation $x = de_s/dt$, we get the general form of the solution

$$e_s(t) = \frac{\alpha}{\beta} A e^{\frac{\beta}{\alpha} t} + \frac{\mu}{\beta} t + C_2 \ . \tag{2.8}$$

The integration constant C_2 is determined from the original equation (1.10) as follows :

$$\mu t + D_0 = \beta \left(\frac{\alpha}{\beta} A e^{\frac{\beta}{\alpha} t} + \frac{\mu}{\beta} t + C_2 \right) - \alpha \left(\frac{\beta}{\alpha} \frac{\alpha}{\beta} A e^{\frac{\beta}{\alpha} t} + \frac{\mu}{\beta} \right) = \mu t + \beta C_2 - \frac{\alpha\mu}{\beta} \ , \tag{2.9}$$

i.e.

$$C_2 = \frac{1}{\beta} D_0 + \frac{\alpha\mu}{\beta^2} \ , \tag{2.10}$$

so that we obtain

$$e_s(t) = \frac{\alpha}{\beta} A e^{\frac{\beta}{\alpha} t} + \frac{\mu}{\beta} t + \frac{1}{\beta} D_0 + \frac{\alpha\mu}{\beta^2} \ . \tag{2.11}$$

This solution (2.11) is just identical with the Flood–Garber special solution (1.10) with (1.12) expect for the first term of the right-hand side in (2.11).

This general form of the solution has also been given in the first paper (1984) [1] by Flood and Garber. However, after this first paper, they had not paid attention to the A-term, for example, in their second paper (1996) [5], and also nobody has been aware of the importance.

Hereafter, we redefine the unknown parameter A as $(\alpha/\beta)A \to Ae^{-(\beta/\alpha)t_c}$ according to the notation by Flood and Garber [1]. Then, the general solution (2.11) can be rewritten as

$$e_s(t) = A e^{\frac{\beta}{\alpha}(t-t_c)} + \frac{1}{\beta} D(t) + \frac{\alpha\mu}{\beta^2} \ . \tag{2.12}$$

Let us go on discussing the role of the A-term in the model below.

Note that the domestic credit function $D(t)$ does not include the exponential term, and it is still given by

$$D(t) = \mu t + D_0 \ . \tag{2.13}$$

We illustrate behaviors of $M(t)$ and $R(t)$ under the assumption of a linear $D(t)$ in Fig. 2. Those behaviors are identical for the cases with the exponential term and the conventional case. On the other hand, the behavior of the shadow exchange rate $e_s(t)$ in the extraordinary solution is clearly different from the conventional one.

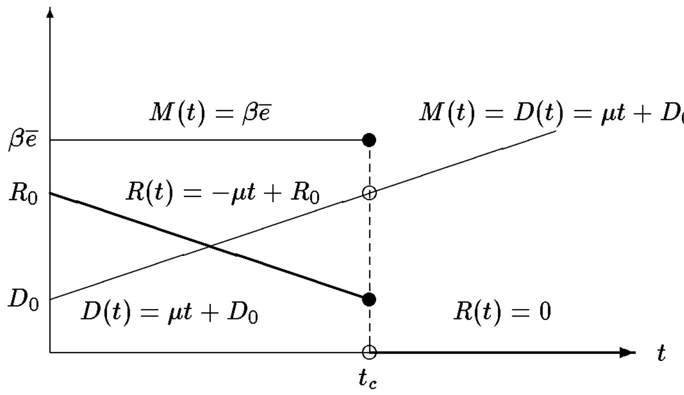

Fig. 2. Behaviors of $M(t)$ and $R(t)$ under the assumption of a linear $D(t)$.

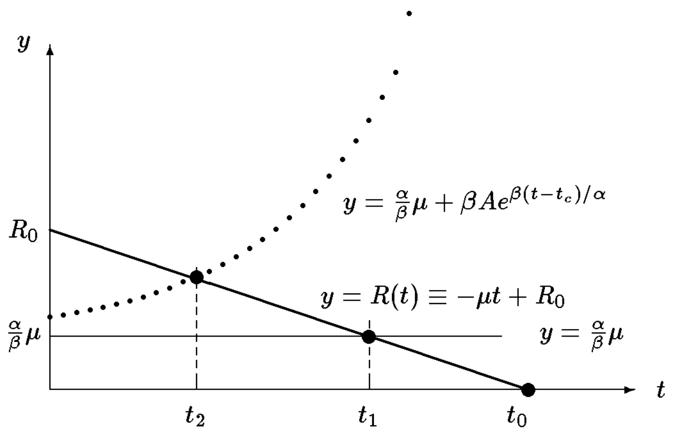

Fig. 3. How to determine the collapse time t_c. The collapse time t_c is given by the intersection point of the functions $y = D(t)$ (a bold solid line) and $y = (\alpha/\beta)\mu + \beta A e^{\beta(t-t_c)/\alpha}$ (a thin solid line for $A = 0$ and a dotted line for $A \neq 0$).

The boundary condition $e_s(t_c) = \bar{e}$ gives

$$\bar{e} = e_s(t_c) = A + \frac{1}{\beta} D(t_c) + \frac{\alpha\mu}{\beta^2} \ , \tag{2.14}$$

so that

$$R(t_c) = \beta A + \frac{\alpha\mu}{\beta} \ , \tag{2.15}$$

where we have used the relation $R(t_c) = \beta\bar{e} - D(t_c)$. Differently from the special case with a linear $e_s(t)$, (1.14), we cannot know the collapse time t_c, because the value of A is unknown in $t < t_c$. We illustrate this situation in Fig. 3. In Fig. 3, the value of the foreign reserves $R(t)$ is known prior to the collapse, and the value $(\alpha/\beta)\mu$ is also known. Since we know the values $R(t_c)$, α, β and μ, if $A = 0$, we can determine the value of t_c from Eq. (2.14) (the value of $t = t_1$ in Fig. 3). However, if $A \neq 0$, we cannot predict the collapse time t_c from (2.14) ($t = t_2$ in Fig. 3). The true collapse time t_c ($t = t_2$ in Fig. 3) always satisfies

$$t_2 < t_1 < t_0 \ , \tag{2.16}$$

where t_1 is the "collapse time" predicted in the conventional Flood–Garber model (i.e. in the case of $A = 0$) and t_0 is the "collapse time" without the concept of the shadow exchange rate (i.e. the time at which the foreign reserves $R(t)$ is exhausted).

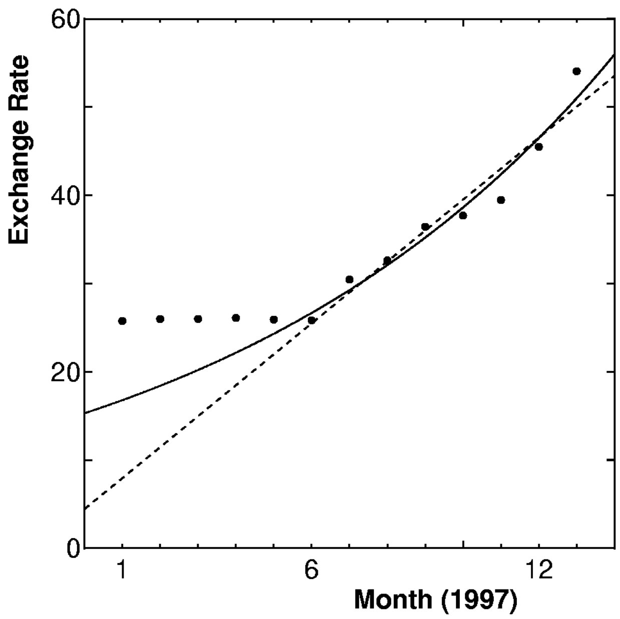

Fig. 4　Behavior of the exchange rate of Baht versus US dollar (e =Baht/US dollar) from January 1997 to January 1998. The dots denote monthly average counter rates quoted by commercial banks (average selling rates), which are quoted from Global Macroeconomic and Financial Policy Site by Nouriel Roubini [6]. In Fig. (a) and (b), the behaviors of $e(t)$ from June 1997 to January 1998 are fitted by a linear function and an exponential function, respectively, and the correlation coefficients $|r|$ are $|r| = 0.967$ and $|r| = 0.982$, respectively.

As an example, we show the behavior of the exchange rate of Baht (the unit of the Thailand currency) from January 1997 to January 1998 in Fig. 4. (Thailand abandoned the (actual) fixed exchange rate regime on July 1997.) The behavior of $e(t)$ from June is fitted by a liner function $e(t) = c_1 t + c_0$ (dashed line) and an exponential function $e(t) = ae^{bt}$ (solid line). (For simplicity, we have assumed that the term $\mu t/\alpha$ is negligibly small compared with the exponential increase in the latter case.) We obtain the best fit parameters $c_1 = 3.50$ and $c_0 = 4.44$ with the correlation coefficient $|r| = 0.967$ for the linear fit, and the best fit parameters $a = 15.3$ and $b = 0.0927$ with $|r| = 0.982$ for the exponential fit. Since the difference of the value $|r|$ in the exponential fit from that in the linear fit is not so large, we cannot conclude that the exponential fit is favorable to the data compared with the linear fit. However, at least, we can conclude that the possibility that the shadow exchange rate $e_s(t)$ contains an exponentially increasing term $Ae^{\beta t/\alpha}$ cannot be ruled out.

3. Can we avoid such the A-term?

As seen in the previous section, we cannot predict the value of the collapse time t_c because of the presence of the A-term. Flood and Garber have put a linearization assumption (1.11), and they have got a linear solution (1.15). However, this assumption cannot be justified at all.

First, let us show that the A-term inevitably appears independently of the assumption on the structure of the domestic credit function $D(t)$, although in the previous section we showed that the A-term appears in the case of a linear $D(t)$.

[Theorem] In general, if a function $e_1(t)$ is a solution of the Flood–Garber differential equation (1.10) [(1.1)], then the following function

$$e_s(t) = e_1(t) + Ae^{\frac{\beta}{\alpha}(t-t_c)} , \tag{3.1}$$

is also a solution of the equation (1.10).

The theorem (3.1) can readily be understood as follows: the function $e_1(t)$ satisfies

$$D(t) = \beta e_1(t) - \alpha \frac{de_1(t)}{dt} , \tag{3.2}$$

so that we can see

$$\beta e_s(t) - \alpha \frac{de_s(t)}{dt} = \beta \left[e_1(t) + Ae^{\frac{\beta}{\alpha}(t-t_c)} \right] - \alpha \left[\frac{de_1(t)}{dt} + \frac{\beta}{\alpha} Ae^{\frac{\beta}{\alpha}(t-t_c)} \right] = \beta e_1(t) - \alpha \frac{de_1(t)}{dt} = D(t) , \tag{3.3}$$

Therefore, the Flood–Garber differential equation (1.10) always has an uncertainty of the A-term. Only the structure of $e_1(t)$ is determined by the structure of $D(t)$, while the value of A is completely independent of the structure of $D(t)$.

The differential equation (1.1) has been derived from the postulations (1.2), (1.3) and (1.4). Does the origin of the A-term originate in the assumption of the linearization (1.3)? The answer is "No". The origin originates in a more fundamental property of the differential equation.

Usually, if we want to predict a future value of a quantity x, we use a differential equation for the quantity x. For example, in order to predict the value of $e_s(t)$, we use a first order differential equation

$$\frac{de_s(t)}{dt} = f[e_s(t), D(t)] \ . \tag{3.4}$$

Then, the solution of $e_s(t)$ inevitably includes one integration constant C (i.e. A). (If the equation is that of a second order, the solution includes two integration constants, and so on.) In order to fix the integration constant, we must impose a boundary condition (an initial condition) on the solution. In the present case, we cannot use a boundary condition at $t = 0$, because we do not know the value of $e_s(t)$ at $t < t_c$. We can use only the boundary condition at $t = t_c$, $e_s(t_c) = \bar{e}$. This is sufficient to fix the value of the integration constant C (or A) for the given value t_c, and no more. In other words, the boundary condition $e_s(t_c) = \bar{e}$ cannot fix the value of t_c, because the integration constant is still unknown.

If we want to fix the value of the integration constant, we must introduce another boundary condition into the model. For example, we require that the asymptotic behavior of $e_s(t)$ $(= e(t))$ must be

$$\frac{1}{e}\frac{de}{dt} \sim \frac{1}{D}\frac{dD}{dt} \ , \tag{3.5}$$

for a sufficiently large t, i.e. at $t \to \infty$. Then , we obtain $A = 0$. The requirement (3.5) leads to the same result $A = 0$ as that in the case with the requirement (1.11) by Flood and Garber. Note that, although the additional requirement (1.11) on the solution of the differential equation cannot be justified mathematically, the use of boundary conditions is a usual technique to fix the integration constants. Then, the problem is whether those boundary conditions are reasonable or not from the phenomenological point of view.

4. Conclusion

The Flood–Garber equation has an extraordinary solution which gives an exponential increase of the shadow exchange rate $e_s(t)$, even if the domestic credit function $D(t)$ is linear. In the Flood and Garber model, the shadow exchange rate $e_s(t)$ is assumed as a linear function in t, so that the collapse time t_c can be predicted by using observed quantities at $t < t_c$. However, such the additional requirement (1.11) to linearize the solution cannot be justified, so that we cannot predict the collapse time t_c any longer, because the coefficients of the exponential term cannot be determined by quantities observed prior to the collapse (i.e. at $t < t_c$), but it can be done only by posteriorly-observed quantities (i.e. at $t > t_c$).

In the present paper, it has been pointed out that the appearance of such an uncertainty term does not originate in the structure of the domestic credit function $D(t)$ and/or in the form of the Flood–Garber equation, but in a more fundamental property of the differential equation. The unknown parameter A corresponds to the integration constant in the first order differential equation. In general, the integration constants cannot be determined from parameters given in the differential equation. Those can be determined only by boundary conditions. We want to predict the collapse time t_c, but we do not know the value of $e_s(t)$ at $t < t_c$, so that we cannot use the value $e_s(0)$ at $t = 0$ as a boundary condition. What we can use as the boundary condition is only one, i.e. the value $e_s(t)$ at $t = t_c$, $e_s(t_c) = \bar{e}$. The condition $e_s(t_c) = \bar{e}$ can fix the value of the integration constant when the value of t_c is given, but, in other words, it means that we cannot use it as a condition to predict the value of t_c. Thus, this problem is independent of the structure of $D(t)$ and/or the explicit form of the Flood–Garber equation. This originates in a more fundamental property of the differential equation. We can never predict the value of the collapse time t_c in principle, as far as we use the method of the differential equation.

If we adhere to the prediction of t_c with use of the differential equation method, we must seek for another boundary condition except for that at $t = t_c$. Only a possible candidate is one at $t \to \infty$. However, in order to find what explicit condition at $t \to \infty$ is reasonable, we will need a more careful study of the macroeconomic (financial) system.

Acknowledgment

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[6] Global Macroeconomic and Financial Policy Site by Nouriel Roubini at the Stern School of Business, New York University; Web address: http:// www.stern.nyu.edu/globalmacro/.