

# Charged Lepton Mass Relations in a SUSY Scenario

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## Abstract

The observed charged lepton masses satisfy the relations  $K \equiv (m_e + m_\mu + m_\tau)/(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2 = 2/3$  and  $\kappa \equiv \sqrt{m_e m_\mu m_\tau}/(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^3 = 1/486$  with great accuracy. These parameters are given as  $K = (\text{Tr}[\Phi\Phi])/(\text{Tr}[\Phi])^2$  and  $\kappa = \det\Phi/(\text{Tr}[\Phi])^3$  if the charged lepton masses  $m_{ei}$  are given by  $m_{ei} \propto \sum_k \Phi_i^k \Phi_k^i$  where  $\Phi$  is a U(3)-family nonet scalar. Simple scalar potential forms to realize the relations have been already proposed in non-supersymmetric scenarios, but the potential forms are not stable against the renormalization group effects. In this paper, we examine supersymmetric scenarios to make the parameters  $K$  and  $\kappa$  stable against the effects, and show possible simple superpotential forms for the relations.

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## 1 Introduction

It is well known that the charged lepton mass relation [1]

$$K \equiv \frac{m_e + m_\mu + m_\tau}{\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau}}^2 = \frac{2}{3}, \quad (1.1)$$

is excellently satisfied by the observed charged lepton masses (pole masses) [2],  $K(m_{ei}^{obs}) = (2/3) \times (0.999989 \pm 0.000014)$ . However, this excellent coincidence causes a confusion because the “mass” in the relation derived in a field theoretical model means, in general, the “running” mass, instead of the “pole” mass.

The deviation of  $K(m_{ei}^{running})$  from  $K(m_{ei}^{pole})$  is caused by a logarithmic term  $\log(\mu/m_{ei})$  in the radiative QED correction [3]

$$m_{ei}(\mu) = m_{ei} \left\{ 1 - \frac{\alpha(\mu)}{\pi} \left( 1 + \frac{3}{4} \log \frac{\mu^2}{m_{ei}^2} \right) \right\}. \quad (1.2)$$

For this problem, Sumino [4] has proposed an attractive mechanism by introducing family gauge bosons  $A_i^j$  with mass  $(M_i^j)^2 \propto (m_{ei} + m_{ej})$ : the logarithmic term in the radiative QED correction is canceled by a logarithmic term  $\log(\mu/M_{ii})$  due to the family gauge bosons.

Recently, another mass relation for the charged leptons [5]

$$\kappa \equiv \frac{\sqrt{m_e m_\mu m_\tau}}{(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^3} = \frac{1}{2 \cdot 3^5} = \frac{1}{486}, \quad (1.3)$$

was proposed. The relation is satisfied by the observed charged lepton masses with accuracy  $\sim 10^{-4}$ .

Let us assume that the charged lepton masses  $m_{ei}$  are given by

$$m_{ei} = k_e \sum_k \langle \Phi \rangle_i^k \langle \Phi \rangle_k^i, \quad (1.4)$$

where  $\langle \Phi \rangle$  is a vacuum expectation value (VEV) of a U(3)-family nonet scalar. The form in Eq. (1.4) is understood from a seesaw scenario [6]. Then, the mysterious mass relations in Eqs. (1.1) and (1.3) can be expressed by somewhat intuitive forms

$$K = \frac{\text{Tr}[\langle \Phi \rangle \langle \Phi \rangle]}{(\text{Tr}[\langle \Phi \rangle])^2} = \frac{2}{3}, \quad (1.5)$$

and

$$\kappa = \frac{\det \langle \Phi \rangle}{(\text{Tr}[\langle \Phi \rangle])^3} = \frac{1}{486}. \quad (1.6)$$

In a non-supersymmetric (non-SUSY) scenario, the relation in Eq. (1.5) has been derived by assuming a scalar potential with a simple form [7]

$$V_K = \mu^2 [\Phi \Phi] + \lambda [\Phi \Phi] [\Phi \Phi] + \lambda' [\Phi_8 \Phi_8] [\Phi]^2. \quad (1.7)$$

Here and hereafter, for convenience, we denote  $\text{Tr}[A]$  as  $[A]$  simply. The scalar  $\Phi_8$  is an octet component of the nonet scalar  $\Phi$ :

$$\Phi_8 \equiv \Phi - \frac{1}{3} [\Phi] \mathbf{1}. \quad (1.8)$$

In Ref. [5], the relation in Eq. (1.6) has been derived by assuming another simple scalar potential form

$$V_\kappa = \lambda' \left( [\Phi_8 \Phi_8 \Phi_8 \Phi_8] + [\Phi_8 \Phi_8 \Phi_8] [\Phi] + [\Phi_8 \Phi_8] [\Phi]^2 + \frac{1}{3^4} [\Phi]^4 - \frac{1}{4} [\Phi_8 \Phi_8] [\Phi_8 \Phi_8] \right). \quad (1.9)$$

Once the relations are obtained for the running masses as above, the Sumino mechanism may ensure the same relations hold also for the pole masses. However, it is not the end of the story: we should also worry about the renormalization group (RG) effects at the high energy scale. As well known, the scalar potentials in non-SUSY models are not stable against the RG evolution. This means that, in order to get the above simple forms at the scale where the field  $\Phi$  is integrated out, the potential forms at the cutoff scale must be complicated or fine-tuned.

In this paper, we consider SUSY models to avoid the problem and give possible superpotential forms which give  $K$  in Eq. (1.5) and  $\kappa$  in Eq. (1.6).

We note that, in a SUSY scenario, the original Sumino mechanism does not work as the vertex corrections with the family gauge boson is suppressed. In addition, in the original model, in order to give the minus sign for the cancellation, the charged leptons ( $e_L, e_R$ ) are assigned to  $(\mathbf{3}, \mathbf{3}^*)$  of the family symmetry  $U(3)$ . Therefore, the original model is not a conventional  $U(3)$  family model with no anomaly in the standard model sector. In Ref. [8], we show that this problem is avoided by introducing family gauge bosons with inversely hierarchical masses  $(M_i^j)^2 \propto (1/m_{ei} + 1/m_{ej})$ , which works in a SUSY scenario, although the cancellation holds only approximately in this case. We assume this modified version of the Sumino mechanism to explain the coincidence between  $K(m_{ei}^{running})$  and  $K(m_{ei}^{pole})$ .

This paper is organized as follows. We give simple superpotentials for the relations of  $K$  in Eq. (1.5) in Sec. 2 and of  $\kappa$  in Eq. (1.6) in Sec. 3, respectively. In Sec. 2, we discuss the stability of the obtained mass relation by applying the discussion given in Ref. [9] for the stability of the effective couplings against the RG effects in a context of the SUSY grand unified theory, to our setup. The Sec. 4 is devoted for the concluding remarks.

## 2 $K$ relation in SUSY scenario

In this section, we construct a SUSY model for the relation of  $K$  in Eq. (1.5).

We assume the following superpotential:

$$W_K = \frac{1}{2}\mu_1\phi_1^2 + \frac{1}{2}\mu_2\phi_2^2 + \mu_3\phi_1\phi_2 + \mu[\Phi\Phi] + \lambda_1[\Phi_8\Phi_8]\phi_1 + \lambda_2[\Phi]^2\phi_2, \quad (2.1)$$

where  $\phi_1$  and  $\phi_2$  are  $U(3)$ -family singlet scalars. Then, we obtain the following three equations:

$$0 = \frac{\partial W_K}{\partial \phi_1} = \mu_1\phi_1 + \mu_3\phi_2 + \lambda_1[\Phi_8\Phi_8], \quad (2.2)$$

$$0 = \frac{\partial W_K}{\partial \phi_2} = \mu_2\phi_2 + \mu_3\phi_1 + \lambda_2[\Phi]^2, \quad (2.3)$$

$$0 = \frac{\partial W_K}{\partial \Phi} = 2\mu\Phi + \lambda_1\phi_1 \left( 2\Phi - \frac{2}{3}[\Phi]\mathbf{1} \right) + \lambda_2 2\phi_2[\Phi]\mathbf{1}. \quad (2.4)$$

Eqs. (2.2) and (2.3) lead to

$$\phi_1 = \frac{1}{\mu_3^2 - 4\mu_1\mu_2} \{ \lambda_1\mu_2[\Phi_8\Phi_8] - \lambda_2\mu_3[\Phi]^2 \}, \quad (2.5)$$

and

$$\phi_2 = \frac{1}{\mu_3^2 - 4\mu_1\mu_2} \{ \lambda_2\mu_1[\Phi]^2 - \lambda_1\mu_3[\Phi_8\Phi_8] \}, \quad (2.6)$$

respectively.

We assume that  $\mu_1$  and  $\mu_2$  are negligibly small compared with  $\mu_3$ . Then, in the limit of  $\mu_1/\mu_3 \rightarrow 0$  and  $\mu_2/\mu_3 \rightarrow 0$ , we obtain

$$\phi_1 = -\frac{\lambda_2}{\mu_3}[\Phi]^2, \quad \phi_2 = -\frac{\lambda_1}{\mu_3}[\Phi_8\Phi_8]. \quad (2.7)$$

When we substitute the VEVs in Eq. (2.7) into  $F$ -flatness condition in Eq. (2.4), we obtain a VEV relation

$$\left( \mu - \frac{\lambda_1\lambda_2}{\mu_3}[\Phi]^2 \right) \Phi - \frac{\lambda_1\lambda_2}{\mu_3} \left( [\Phi\Phi] - \frac{2}{3}[\Phi]^2 \right) [\Phi]\mathbf{1} = 0. \quad (2.8)$$

In order that there is a VEV value  $\langle \Phi \rangle \neq \mathbf{1}$ , both the coefficients of  $\Phi$  and  $\mathbf{1}$  must be zero, so that we obtain the following relations

$$\mu - \frac{\lambda_1\lambda_2}{\mu_3}[\Phi]^2 = 0, \quad (2.9)$$

and

$$[\Phi\Phi] - \frac{2}{3}[\Phi]^2 = 0. \quad (2.10)$$

The relation in Eq. (2.9) plays a role of fixing the scale of VEV of  $\Phi$ , dependently on the parameters  $\mu$ ,  $\mu_3$ ,  $\lambda_1$  and  $\lambda_2$ . The VEV relation in Eq. (2.10) is just the one which we desired. Note that the VEV relation is independent of the potential parameters  $\mu_1$ ,  $\mu_2$ ,  $\mu_3$ ,  $\mu$ ,  $\lambda_1$  and  $\lambda_2$ .

Here, we give a discussion on the stability of the VEV relations. We note that the superpotential in Eq. (2.1) is not general, as the scalar potential in Eq. (1.7). For example, the term  $\mu[\Phi\Phi] = \mu([\Phi_8\Phi_8] + \frac{1}{3}[\Phi]^2)$  is a special combination of the two irreducible terms, which have different RG evolutions from each other. Then, at first glance, this SUSY model might seem to suffer from the same problem as the non-SUSY models. In fact, it is not the case [9]: the nonrenormalization theorem ensures that the effective leptonic Yukawa couplings runs in the same way as in the minimal SUSY standard model (MSSM). An important point is that, although the VEV  $\langle \Phi \rangle$  actually runs also in the SUSY model, the running is caused only by the wave function renormalization due to the renormalization of the Kahler potential since the superpotential is not renormalized. It is straightforward to see that this running of the VEV cancels the wave function renormalization of the field  $\Phi$  in the coupling  $l\Phi\Phi e h_d$ , which gives effectively the leptonic Yukawa interactions. We can understand this cancellation as follows. The wave function renormalization is nothing but just the renaming of the fields, which does not affect the ‘‘physics’’ that the effective leptonic Yukawa interaction is given by the coupling with the ‘‘physical’’ VEV  $\langle \Phi \rangle$ . This is analogous to the fact the physical length is independent of the measure. In this way, we understand that in the SUSY models, the effective leptonic Yukawa

coupling runs independently of the running of the VEV  $\langle \Phi \rangle$  as far as the VEV is determined by the superpotential, which is protected by the nonrenormalization theorem. This ensures that once we obtain the simple superpotential for the relation of  $K$  in Eq. (1.5) at the cutoff scale in some way, the relation holds also at the scale where the field  $\Phi$  is integrated out, even though the superpotential at the scale is corrected to become complicated due to the RG evolution. Thus, we conclude that in this SUSY model with the help of the modified Sumino mechanism, we may explain the relation  $K = 2/3$  holds for the pole masses, assuming the relation is obtained for the running masses at the cutoff scale of the SUSY model.

### 3 The $\kappa$ relation in SUSY scenario

In this section, we construct a SUSY model for the relation of  $\kappa$  in Eq. (1.6).

Analogously to the previous section, we assume the following superpotential with a simple form

$$W_\kappa = \mu_{AB}[AB] + \lambda_A \{[\Phi\Phi A] + \alpha[\Phi\Phi][A]\} + \lambda_B \{[\Phi_8\Phi_8 B] + \beta[\Phi_8\Phi_8][B]\}, \quad (3.1)$$

where  $A$  and  $B$  are U(3)-family nonet scalars and we set the mass terms  $\mu_A[AA]$  and  $\mu_B[BB]$  negligible compared with  $\mu_{AB}[AB]$ , as in the previous section. The  $\lambda_B$  terms can be re-written as

$$\lambda_B \left\{ [\Phi\Phi B] - \frac{2}{3}[\Phi][\Phi B] + \beta[\Phi\Phi][B] + \frac{1}{9}(1 - 3\beta)[\Phi]^2[B] \right\}. \quad (3.2)$$

We can obtain the following three VEV relations:

$$0 = \frac{\partial W_\kappa}{\partial A} = \mu_{AB}B + \lambda_A(\Phi\Phi + \alpha[\Phi\Phi]\mathbf{1}), \quad (3.3)$$

$$0 = \frac{\partial W_\kappa}{\partial B} = \mu_{AB}A + \lambda_B \left\{ \Phi\Phi - \frac{2}{3}[\Phi]\Phi + \beta[\Phi\Phi]\mathbf{1} + \frac{1}{9}(1 - 3\beta)[\Phi]^2\mathbf{1} \right\}, \quad (3.4)$$

$$0 = \frac{\partial W_\kappa}{\partial \Phi} = \lambda_A \{(\Phi A + A\Phi) + 2\alpha[A]\Phi\} \\ + \lambda_B \left\{ (\Phi B + B\Phi) - \frac{2}{3}[\Phi]B - \frac{2}{3}[\Phi B]\mathbf{1} + 2\beta[B]\Phi + \frac{2}{9}(1 - 3\beta)[\Phi][B]\mathbf{1} \right\}. \quad (3.5)$$

By substituting Eqs. (3.3) and (3.4) into Eq. (3.5), we obtain a VEV relation

$$0 = -\frac{\partial W_\kappa}{\partial \Phi} = \frac{\lambda_A \lambda_B}{\mu_{AB}} \left\{ 4\Phi\Phi\Phi - \frac{4}{3}[\Phi]\Phi\Phi + 4(\alpha + \beta + 3\alpha\beta)[\Phi\Phi]\Phi \right. \\ \left. + \left( \frac{2}{9}(1 + 3\alpha(1 - 3\beta)) - \frac{4}{3}\alpha - \frac{2}{3}\beta \right) [\Phi]^2\Phi - \frac{2}{3}[\Phi\Phi\Phi]\mathbf{1} \right\}$$

$$+ \left( -\frac{4}{3}\alpha + \frac{2}{9}(1+3\alpha)(1-3\beta) \right) [\Phi][\Phi\Phi] \mathbf{1} \Big\}. \quad (3.6)$$

Since, for an arbitrary  $3 \times 3$  matrix  $A$ , we know relations

$$AAA = [A]AA + \frac{1}{2} ([AA] - [A]^2) A + \det A \mathbf{1}, \quad (3.7)$$

and

$$[AAA] = 3\det A + \frac{3}{2}[AA][A] - \frac{1}{2}[A]^3, \quad (3.8)$$

we can re-write the relation in Eq. (3.6) as follows:

$$0 = \left\{ 2\det\Phi - \frac{1}{27}(5 + 12(\alpha + \beta + 3\alpha\beta)) [\Phi]^3 \right\} \mathbf{1} + (\Phi \text{ and } \Phi\Phi \text{ terms}), \quad (3.9)$$

Here, we have already rewritten  $[\Phi\Phi]$  as

$$[\Phi\Phi] = K_0 + \frac{2}{3}[\Phi]^2, \quad (3.10)$$

with  $K_0 = 0$  from the result in Eq. (2.10). Since  $\Phi$ - and  $\Phi\Phi$ -terms are counterbalanced by adding  $\mu[\Phi\Phi]$  and  $\lambda[\Phi\Phi\Phi]$  to the original superpotential in Eq. (3.1), our interest is only in the coefficient of the unit matrix  $\mathbf{1}$  in Eq. (3.9).

Thus, we can finally re-write the relation in Eq. (3.9) only with  $[\Phi]^2$  and  $\det\Phi$  and, thereby, we obtain a relation on  $\kappa$

$$\kappa = \frac{\det\Phi}{[\Phi]^3} = +\frac{1}{54} \{5 + 12(\alpha + \beta + 3\alpha\beta)\}. \quad (3.11)$$

Since we want to reproduce the numerical result in Eq. (1.6), the condition is given by

$$11 + 27(\alpha + \beta + 3\alpha\beta) = 0. \quad (3.12)$$

We suppose that the coefficients  $\alpha$  and  $\beta$  are given as simple rational numbers because we consider that the relation in Eq. (1.6), as well as that in Eq. (1.5), has a connection with the fundamental physics. We find simple solutions of the condition in Eq. (3.12);

$$(\alpha, \beta) = \left(-\frac{1}{9}, -\frac{4}{9}\right), \quad \text{and} \quad (\alpha, \beta) = \left(-\frac{4}{9}, -\frac{1}{9}\right), \quad (3.13)$$

*i.e.*

$$W_\kappa = \mu[AB] + \lambda_A \left\{ [\Phi\Phi A] - \frac{1}{9}[\Phi\Phi][A] \right\} + \lambda_B \left\{ [\Phi_8\Phi_8 B] - \frac{4}{9}[\Phi_8\Phi_8][B] \right\}, \quad (3.14)$$

or  $W_\kappa$  with  $\alpha \leftrightarrow \beta$ .

Although the form of  $W_\kappa$  is not general at all, the nonrenormalization theorem again protect the effective leptonic Yukawa couplings against the RG effect, as discussed in the previous section.

## 4 Concluding remarks

In conclusion, on the basis of SUSY framework, we have found superpotential forms  $W_K$  and  $W_\kappa$  with simple coefficients, which lead to the charged lepton mass relations in Eqs. (1.1) and (1.3), i.e. those in Eqs. (1.5) and (1.6), respectively. Of course, the potential forms which has been proposed in this paper is not unique. Those potential forms have been obtained under a guiding principle that the coefficients in the potential should be given by rational numbers as simple as possible. At first glance, the relations derived from such tuned superpotentials might seem to be destabilized by the RG effects even in the SUSY models. We showed that, however, the relations are stable against the effects by applying the discussion given in Ref. [9] in a context of the SUSY grand unified theory to our set up. Thus, we conclude that, once the simple superpotential forms are realized (possibly at the cutoff scale) in some way in our SUSY models, with the help of the modified Sumino mechanism [8], the desired mass relations holds also for the pole masses.

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