Charged Lepton Mass Formula
-- Development and Prospect --

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4 Summary
1 Beginning of the charged lepton mass formula

It is generally considered that masses and mixings of the quarks and leptons will obey a simple law of nature, so that we expect that we will find a beautiful relation among those values. It is also considered that the mass matrices of the fundamental particles will be governed by a symmetry.
My dream is to understand masses and mixings from a symmetry, i.e. not from adjustable parameters, but from Clebsch-Gordan-like coefficients.

For example, in 1971, I tried to understand the Cabibbo mixing from a U(3) symmetry.
The Cabibbo Angle and the Structure of Urbaryons

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By introducing the group $U(4)$ into the world of leptons, we discuss the group theoretical structure of the Hiroshima model, in which urbaryons are assumed to be compound systems of a massive Dirac particle $B$ and lepton–antilepton clounds. Under some assumptions for the properties of the $B$–particle, we can obtain $\sin \theta = (\sqrt{3}-1)/2\sqrt{2} = 0.259$ and $\cos \theta = (\sqrt{3}+1)/2\sqrt{2} = 0.966$ as the value of Cabibbo angle $\theta$ from kinematical discussions.
Hiroshima model

T.Hayashi, YK, S.Ogawa, PTP (1968)

\[ u = \left\langle \left( \nu_e \bar{\nu}_e \cos \theta + \nu_\mu \bar{\nu}_\mu \sin \theta \right) \right\rangle B \]

\[ c = \left\langle \left( -\nu_e \bar{\nu}_e \sin \theta + \nu_\mu \bar{\nu}_\mu \cos \theta \right) \right\rangle B \]

\[ d = \left\langle \left( e^- \bar{\nu}_e \right) B \right\rangle \]

\[ s = \left\langle \left( \mu^- \bar{\nu}_\mu \right) B \right\rangle \]

Semileptonic decays \( d \rightarrow u \) and \( s \rightarrow u \) are understood from \( e^- \rightarrow \nu_e \) with the factor \( \cos \theta \) and from \( \mu^- \rightarrow \nu_\mu \) with the factor \( \sin \theta \), respectively.

Nonleptonic decay \( s \rightarrow d \) can be understood from \( \mu^- \bar{\nu}_\mu \rightarrow e^- \bar{\nu}_e \) with the factor 1 without assuming the so-called penguin diagram.
The basis idea was analogy to the hadronic $\pi^0 - \eta^0 - \sigma^0$ mixing where we assume $(\nu_e, \nu_\mu, \ell) \sim 3$ of U(3), and 3 quarks are members of $8+1$ of U(3);

Eigenstates of U-spin

\[
\begin{align*}
\pi &= \frac{1}{\sqrt{2}}(\nu_e\bar{\nu}_e - \nu_\mu\bar{\nu}_\mu) \\
\eta &= \frac{1}{\sqrt{6}}(\nu_e\bar{\nu}_e + \nu_\mu\bar{\nu}_\mu - 2\ell\bar{\ell}) \\
\sigma &= \frac{1}{\sqrt{3}}(\nu_e\bar{\nu}_e + \nu_\mu\bar{\nu}_\mu + \ell\bar{\ell})
\end{align*}
\]

Eigenstates of I-spin

\[
\begin{align*}
\pi &= \frac{1}{\sqrt{2}}(\nu_e\bar{\nu}_e - \nu_\mu\bar{\nu}_\mu) \\
\omega &= \frac{1}{\sqrt{2}}(\nu_e\bar{\nu}_e + \nu_\mu\bar{\nu}_\mu) \\
\phi &= \ell\bar{\ell}
\end{align*}
\]

\[
\begin{align*}
u_0 &= (\pi B) & c_0 &= (\eta B) \\
d_0 &= (\ell\nu_e B) & s_0 &= (\ell\nu_\mu B)
\end{align*}
\]

\[
\begin{align*}
u_0 &= (\pi' B) & c &= (\omega' B) \\
d &= (e\nu_e B) & s &= (\mu\nu_\mu B)
\end{align*}
\]

\[
\begin{align*}
\pi' &= \frac{\sqrt{3}}{2}\pi + \frac{1}{2}\omega = \frac{\sqrt{3} + 1}{2\sqrt{2}}\nu_e\bar{\nu}_e - \frac{\sqrt{3} - 1}{2\sqrt{2}}\nu_\mu\bar{\nu}_\mu
\end{align*}
\]
The model was strained and tricky. Therefore, the paper was rejected to be published by PTP. I think that the rejection was reasonable.

In 1978, Harari et al. have proposed a model based on a permutation symmetry $S_3$:

H. Harari, H. Haut and J. Weyers, PL 78B (1978) 459

They have derived the Cabibbo angle

$$\sin \theta_c = \frac{\sqrt{3} - 1}{2\sqrt{2}}, \quad \cos \theta_c = \frac{\sqrt{3} + 1}{2\sqrt{2}}$$

Their work was very beautiful, so that I was a little discouraged.
In 1982, I have proposed a charged lepton mass relation

$$m_e + m_\mu + m_\tau = \frac{2}{3} \left( \sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau} \right)^2$$

(1.1)

It is well-known that the observed charged lepton mass spectrum satisfies the relation with remarkable precision.

[YK, LNC (1982); PLB (1983)]

The mass formula (1.1) is invariant under any exchange $\sqrt{m_i} \leftrightarrow \sqrt{m_j}$.

This suggests that a description by $S_3$ may be useful for the mass matrix model.

[S3: Pakvasa and Sugawara (1978); Harari, Haut and Weyers (1978)]

However, this formula was derived from an extension of the $\pi-\eta-\sigma$ mixing model (composite model), and I had never considered an $S_3$ symmetry until 1999.
In the present talk, I would like to demonstrate how the $S_3$ symmetry is promising to understand the masses and mixings. And, I will comment that an $A_4$ symmetry is also promising.

Although we think that there are beautiful relations among quark and lepton masses and mixings, it is hard to see such a symmetry in the quark sector, because the original symmetry will be spoiled by the gluon cloud.

In the present study, I will confine myself to the investigation of the lepton masses and mixings.
2 Leptons under $S_3$

I would like to show that the $S_3$ symmetry is promising not only for understanding of the charged lepton mass relation (1.1), but also for that of the observed neutrino mixing, i.e. the so-called tribimaximal mixing

$$U_{TB} = \begin{pmatrix}
-\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}
\end{pmatrix}$$ (2.1)

[Harrison, Perkins, Scott, PLB (1999); Xing, PLB (2002); Ma, PRL (2003); ]
2.1 Tribimaximal mixing

We define the doublet \((\psi_\pi, \psi_\eta)\) and singlet \(\psi_\sigma\) of \(S_3\):

\[
\begin{pmatrix}
\psi_\pi \\
\psi_\eta \\
\psi_\sigma
\end{pmatrix} =
\begin{pmatrix}
0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
-\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}
\end{pmatrix}
\begin{pmatrix}
\psi_1 \\
\psi_2 \\
\psi_3
\end{pmatrix}
\] (2.2)
If the neutrino mass eigenstates are defined by the relation (2.2)
with the mass hierarchy \( m_{\nu_\eta}^2 < m_{\nu_\sigma}^2 < m_{\nu_\pi}^2 \)
in contrast to the weak eigenstates
\( (\nu_1, \nu_2, \nu_3) = (\nu_e, \nu_\mu, \nu_\tau) \),
then we can obtain the tribimaximal mixing (2.1) because of the relation
\[
\begin{pmatrix}
\nu_e \\
\nu_\mu \\
\nu_\tau
\end{pmatrix}
= \begin{pmatrix}
\nu_1 \\
\nu_2 \\
\nu_3
\end{pmatrix}
= \begin{pmatrix}
-\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}
\end{pmatrix}
\begin{pmatrix}
\nu_\eta \\
\nu_\sigma \\
\nu_\pi
\end{pmatrix}
\]
2.2 Charged lepton mass formula and Higgs potential

The mass formula (1.1) can be understood from a universal seesaw model with 3-flavor scalars \( \phi_i \)

\[
M_f = m_f^L M_F^{-1} m_f^R
\]

For the charged lepton sector, we take

\[ m_L^e = \frac{1}{\kappa} m_R^e = y_e \text{diag}(v_1, v_2, v_3) \]  

(2.5)

where the VEV \( v_i = \langle \phi_i \rangle \) satisfy the relation

\[ v_1^2 + v_2^2 + v_3^2 = \frac{2}{3} (v_1 + v_2 + v_3)^2 \]  

(2.6)

The VEV relation (2.6) means

\[ v_\pi^2 + v_\eta^2 = v_\sigma^2 \]  

(2.7)

because

\[ v_1^2 + v_2^2 + v_3^2 = v_\pi^2 + v_\eta^2 + v_\sigma^2 \]

\[ = 2v_\sigma^2 = 2 \left( \frac{v_1 + v_2 + v_3}{\sqrt{3}} \right)^2 \]  

(2.8)
The Higgs potential which gives the relation (2.7) is, for example, given by

\[ V = \mu^2 \sum_i (\phi_i \phi_i) + \frac{1}{2} \lambda_1 \left( \sum_i (\phi_i \phi_i) \right)^2 + \lambda_2 (\phi_\sigma \phi_\sigma)(\phi_\pi \phi_\pi + \phi_\eta \phi_\eta) + V_{\text{soft sym.br}} \]  

(2.9)

Here, the existence of the $S_3$ invariant $\lambda_2$-term is essential. Note that the VEV relation (2.7) can be obtained independently of the explicit values of the parameter $\lambda_2$. 

YK, PRD73 (2006),
2.3 Yukawa interaction form under $S_3$

The general form of the $S_3$ invariant Yukawa interaction is given by

$$II = \left( y_0 \frac{\bar{\psi}_\pi \psi_\pi + \bar{\psi}_\eta \psi_\eta + \bar{\psi}_\sigma \psi_\sigma}{\sqrt{3}} + y_1 \frac{\bar{\psi}_\pi \psi_\pi + \bar{\psi}_\eta \psi_\eta - 2\bar{\psi}_\sigma \psi_\sigma}{\sqrt{6}} \right) \phi_\sigma$$

$$+ y_2 \left( \frac{\bar{\psi}_\pi \psi_\eta + \bar{\psi}_\eta \psi_\pi}{\sqrt{2}} \phi_\pi + \frac{\bar{\psi}_\pi \psi_\pi - \bar{\psi}_\eta \psi_\eta}{\sqrt{2}} \phi_\eta \right)$$

$$+ y_3 \frac{\bar{\psi}_\pi \phi_\pi + \bar{\psi}_\eta \phi_\eta}{\sqrt{2}} \psi_\sigma + y_4 \frac{\phi_\pi \psi_\pi + \phi_\eta \psi_\eta}{\sqrt{2}}$$

$(2.10)$
For the charged lepton sector, we have already assumed the form

\[
H_e = y_e \left( \bar{\ell} L_1 \phi^d_{L1} E_{R1} + \bar{\ell} L_2 \phi^d_{L2} E_{R2} + \bar{\ell} L_3 \phi^d_{L3} E_{R3} \right) \quad (2.11)
\]

The form (2.11) corresponds to the case

\[
y_0 = y_e, \quad y_1 = 0, \quad y_2 = \frac{1}{\sqrt{3}} y_e, \quad y_3 = y_4 = \sqrt{\frac{2}{3}} y_e \quad (2.12)
\]

in the general form (2.10).

The general study under the $S_3$ symmetry will be found in YK, a preprint hep-ph/0612058. Here, I will skip the details and let us review only a simple example.
Now, I would like to propose an $S_3$-invariant neutrino interaction with a concise form

$$H_\nu = y_\nu \left( \frac{\ell_\pi N_\pi + \ell_\eta N_\eta + \ell_\sigma N_\sigma}{\sqrt{3}} \phi^u_\sigma \right.\left. + \frac{\ell_\pi N_\eta + \ell_\eta N_\pi}{\sqrt{2}} \phi^u_\pi + \frac{\ell_\pi N_\pi - \ell_\eta N_\eta}{\sqrt{2}} \phi^u_\eta \right)$$

(2.13)

Here, we have assumed

the universality of the coupling constants
to the $\phi_\sigma$, $\phi_\pi$ and $\phi_\eta$ terms.
Then, we obtain a simple mass spectrum

\[
m_{\nu 1} = \left( \frac{1}{\sqrt{6}} - \frac{1}{2} \right)^2 m_0^\nu
\]
\[
m_{\nu 2} = \frac{1}{6} m_0^\nu
\]
\[
m_{\nu 3} = \left( \frac{1}{\sqrt{6}} + \frac{1}{2} \right)^2 m_0^\nu
\]

Here, we have used the relation

\[
\nu_\pi^2 + \nu_\eta^2 = \nu_\sigma^2
\]  \hspace{1cm} (2.15)

from the S_3 Higgs potential model.
The present case predicts

\[ R = \frac{\Delta m_{21}^2}{\Delta m_{32}^2} = \frac{4\sqrt{6} - 9}{4\sqrt{6} + 9} = 0.0423 \]  

(2.16)

The predicted value is somewhat larger than

\[ R_{\text{obs}} \equiv \frac{\Delta m_{\text{solar}}^2}{\Delta m_{\text{atm}}^2} = \frac{(7.9^{+0.6}_{-0.5}) \times 10^{-5}\text{eV}^2}{(2.72^{+0.38}_{-0.25}) \times 10^{-3}\text{eV}^2} \]

(2.17)

\[ = (2.9 \pm 0.5) \times 10^{-2} \]

However, the case cannot, at present, be ruled out within three sigma.
If $\nu_\pi / \nu_\eta \neq 0$, $m^\nu_L$ cannot be diagonal on the basis $(\nu_\pi, \nu_\eta, \nu_\sigma)$. The mixing angle $\theta_{\pi \eta}$ of the further rotation between $\nu_\pi - \nu_\eta$ is given by

$$\tan \theta_{\pi \eta} = \frac{\nu_\pi}{\nu_\eta}$$

(2.18)

It is well known that the $2 \leftrightarrow 3$ symmetry is promising for neutrino mass matrix description.


We also assume the $2 \leftrightarrow 3$ symmetry for $\langle \phi^u_i \rangle$, i.e.

$$\nu^u_2 = \nu^u_3$$

, which leads to

$$\langle \phi^u_\pi \rangle = 0$$

(2.19)
Therefore, the present model gives the exact tribimaximal mixing:

\[
\sin^2 2\theta_{23} = 1, \quad \tan^2 \theta_{12} = 1/2, \quad \theta_{13} = 0
\] (2.20)

Note that if we require the \( 2 \leftrightarrow 3 \) symmetry for the fields \( \ell_{Li} = (\nu_{Li}, e_{Li}) \), the symmetry will affect the charged lepton sector, too. Here, we have assumed the \( 2 \leftrightarrow 3 \) symmetry only for \( \langle \phi^u_i \rangle \), not for \( \langle \phi^d_i \rangle \), so that the symmetry does not affect the charged lepton mass matrix.
Summery of the $S_3$ model

• We have assumed a universal seesaw model.
• The $S_3$ symmetry in the Yukawa interactions is strictly unbroken.
• $S_3$ is broken only though the VEV of Higgs scalars $\phi_i$.
• We have required the universality of the coupling constants on the basis $(e_1, e_2, e_3)$ for the charged lepton sector, while those on the basis $(\nu_\pi, \nu_\eta, \nu_\sigma)$ for the neutrino sector.
3  Prospect of the Charged Lepton Mass Formula

3.1 Brannen's speculations

Recently, Brannen has speculated a neutrino mass relation similar to the charged lepton mass relation (1.1):

\[ m_{\nu 1} + m_{\nu 2} + m_{\nu 3} = \frac{2}{3} \left( -\sqrt{m_{\nu 1}} + \sqrt{m_{\nu 2}} + \sqrt{m_{\nu 3}} \right)^2 \]  

(3.1)
Of course, we cannot extract the values of the neutrino mass ratios $m_{\nu_1}/m_{\nu_2}$ and $m_{\nu_2}/m_{\nu_3}$ from the neutrino oscillation data $\Delta m^2_{solar}$ and $\Delta m^2_{atm}$ unless we have more information on the neutrino masses, so that we cannot judge whether the observed neutrino masses satisfy the relation (3.1) or not.
Generally, the masses which satisfy the relations (1.1) and (3.1) can be expressed as

\[ m_{f_i} = (z_{f_i})^2 m_{f_0} \]  \hspace{1cm} (3.2)

where

\[ z_{f1} = \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}} \sin \xi_f \]
\[ z_{f2} = \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}} \sin(\xi_f + \frac{2}{3}\pi) \]  \hspace{1cm} (4.3)
\[ z_{f3} = \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}} \sin(\xi_f + \frac{4}{3}\pi) \]

\[ (z_{f1})^2 + (z_{f2})^2 + (z_{f3})^2 = 1 \]  \hspace{1cm} (3.4)

Then, Brannen has also speculated the relation

\[ \xi_\nu = \xi_e + \frac{\pi}{12} \]  \hspace{1cm} (3.5)
From the observed charged lepton mass values, we obtain

$$\xi_e = \frac{\pi}{4} - \epsilon = 42.7324^\circ \quad (\epsilon = 2.2676^\circ) \quad (3.6)$$

Then, the Brannen relation (3.5) gives

$$\xi_\nu = 57.7324^\circ \quad (3.7)$$

which predicts

$$R = \frac{\Delta m_{21}^2}{\Delta m_{32}^2} = 0.0318 \quad (3.8)$$

The value (3.8) is in good agreement with the observed value 0.029 +/- 0.005.

Therefore, the speculations by Brannen are favorable to the observed neutrino data.
We can understand Brannen's first relation (3.1) by assuming two types of $\phi^f$ i.e. $\phi^u \neq \phi^d$. However, Brannen's second relation (3.5) is hard to be derived from the conventional symmetries. It is an open question that the relation (3.5) is accidental or not.

[YK, hep-ph/0612058]
Besides, Brannen (2006) and G. Rosen (2006) have speculated that the observed value $\xi_e = 42.7324^\circ$ is given by the relation

$$\xi_e = \frac{\pi}{6} + \frac{2}{9} \quad (3.9)$$

In deed, the value $2/9$ is

$$2/9 \text{ rad} = 12.732395^\circ \quad (3.10)$$

It is an amazing coincidence!

However, my opinion is as follows:

Too adhering to this coincidence will again push the formula (1.1) into a mysterious world, so that I do not recommend that you take the relation (3.9) seriously at present.
3.2 From Seesaw to Frogatt-Nielsen

The seesaw model with three SU(2)_L doublet scalars causes the FCNC problem. Therefore, the seesaw model may, for example, be translated into a Frogatt-Nielsen-type model:

\[
H_{\text{eff}} = y_e \bar{e}_L H^d_L \frac{\phi^d \phi^d}{\Lambda_d \Lambda_d} e_R
\]

\[+ y_\nu \bar{\nu}_L H^u_L \frac{\phi^u \nu_R}{\Lambda_u} + y_R \bar{\nu}_R \Phi \nu^*_R \quad (3.11)\]

The argument about flavor structure is essentially unchanged under the present model-changing.
However, the scenario for the symmetry breaking energy scale will be considerably changed:

<table>
<thead>
<tr>
<th>Energy scale of $v_i = \langle \phi_i \rangle$</th>
<th>Seesaw</th>
<th>Frogatt-Nielsen</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^2$ GeV</td>
<td>$10^{19}$ GeV</td>
<td></td>
</tr>
</tbody>
</table>

In the Frogatt-Nielsen-type model, the $S_3$-broken structure of the effective Yukawa coupling constants is given at the Planck mass scale. However, this is no problem, because the formula (1.1) is not so sensitive to the RGE effects as far as the lepton sector is concerned.

[Xing-Zhang PLB (2006), Li-Ma PRD (2006)]
3.3 From $S_3$ to $A_4$

Recently, Ma (UC, Riverside) has explained the observed tribimaximal mixing on the basis of $A_4$. This suggests that the $A_4$ symmetry is also promising as a symmetry of leptons.

Ma, Phys.Rev. D73, 057304 (2006)
• Irreducible representations of $A_4$

When we denote 3 of $A_4$ as

$$\overline{\psi} = \begin{pmatrix} \overline{\psi_1} \\ \overline{\psi_2} \\ \overline{\psi_3} \end{pmatrix} \sim 3, \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \sim 3$$

(3.12)

we can make 1, 1', 1" from 3x3 as follows:

$$(\overline{\psi}\psi)_1 = \frac{1}{\sqrt{3}}(\overline{\psi_1}\psi_1 + \overline{\psi_2}\psi_2 + \overline{\psi_3}\psi_3)$$

$$(\overline{\psi}\psi)_1' = \frac{1}{\sqrt{3}}(\overline{\psi_1}\psi_1 + \overline{\psi_2}\psi_2\omega + \overline{\psi_3}\psi_3\omega^2)$$

$$(\overline{\psi}\psi)_1'' = \frac{1}{\sqrt{3}}(\overline{\psi_1}\psi_1 + \overline{\psi_2}\psi_2\omega^2 + \overline{\psi_3}\psi_3\omega)$$

(3.13)

where

$$\omega = e^{i\frac{2}{3}\pi} = \frac{-1 + i\sqrt{3}}{2}$$

(3.14)
When we define
\[
(\overline{\psi}\psi)_{\sigma} = \frac{1}{\sqrt{3}} (\overline{\psi}_1 \psi_1 + \overline{\psi}_2 \psi_2 + \overline{\psi}_3 \psi_3)
\]
\[
(\overline{\psi}\psi)_{\eta} = \frac{1}{\sqrt{6}} (2\overline{\psi}_1 \psi_1 - \overline{\psi}_2 \psi_2 - \overline{\psi}_3 \psi_3)
\]
\[
(\overline{\psi}\psi)_{\pi} = \frac{1}{\sqrt{2}} (\overline{\psi}_3 \psi_3 - \overline{\psi}_2 \psi_2)
\]

we can express (3.13) as

\[
(\overline{\psi}\psi)_1 = (\overline{\psi}\psi)_{\sigma}
\]
\[
(\overline{\psi}\psi)_{1'} = \frac{1}{\sqrt{2}} \left[ (\overline{\psi}\psi)_{\eta} - i(\overline{\psi}\psi)_{\pi} \right]
\]
\[
(\overline{\psi}\psi)_{1''} = \frac{1}{\sqrt{2}} \left[ (\overline{\psi}\psi)_{\eta} + i(\overline{\psi}\psi)_{\pi} \right]
\]
Therefore, if we define \( (\phi_\pi, \phi_\eta, \phi_\sigma) \) as

\[
\begin{align*}
\phi_1 &= \phi_\sigma \\
\phi_1' &= \frac{1}{\sqrt{2}}(\phi_\eta - i\phi_\pi) \\
\phi_1'' &= \frac{1}{\sqrt{2}}(\phi_\eta + i\phi_\pi)
\end{align*}
\]  \hspace{1cm} (3.17)

we can write an \( A_4 \) invariant Yukawa interaction as follows:

\[
\begin{align*}
(\overline{\psi}\psi)_1 \phi_1 + (\overline{\psi}\psi)_1' \phi_1'' + (\overline{\psi}\psi)_1'' \phi_1' \\
= (\overline{\psi}\psi)_\sigma \phi_\sigma + (\overline{\psi}\psi)_\eta \phi_\eta + (\overline{\psi}\psi)_\pi \phi_\pi
\end{align*}
\]  \hspace{1cm} (3.18)

Thus, we can translate the relations in \( S_3 \) into those in \( A_4 \). The \( A_4 \) symmetry will be also promising in the symmetry of the leptons.
The existence of the $\lambda_2$-term, $\phi_\sigma^2(\phi_\pi^2 + \phi_\eta^2)$, was essential in the $S_3$ Higgs potential model. In an $A_4$ model, the $\lambda_2$-term can be composed as

$$\phi_1 \phi_1 (\phi_1' \phi_1'' + \phi_1'' \phi_1')$$

$$= \phi_\sigma^2 (\phi_\pi^2 + \phi_\eta^2)$$

(3.19)

Thus, the charged lepton mass relation (1.1) can also be derived from the $A_4$ model.
3.4 From non-SUSY to SUSY

Since the formula (1.1) is so exact, we need to find a condition which is protected against large corrections.

Ma has recently proposed a **supersymmetric S₃-invariant Higgs potential**, because supersymmetry is unbroken even after the superfields acquire VEVs:

\[
W = \frac{1}{2}m \left( \phi_\pi^2 + \phi_\eta^2 + \phi_\sigma^2 \right) + \frac{1}{3} \lambda \phi_\sigma^3 + \lambda \phi_\sigma \left( \phi_\pi^2 + \phi_\eta^2 \right)
\]  

(3.20)

which also leads to the relation

\[
\nu_\pi^2 + \nu_\eta^2 = \nu_\sigma^2.
\]
By extending this idea, and by applying a symmetry $\Sigma(81)$ to the model, Ma has proposed a new model which leads to the charged lepton mass relation (1.1):

$$W = \frac{1}{2}m_0\chi_0^2 + \frac{1}{2}m_1\chi_1^2 + \frac{1}{2}m_2\chi_2^2 + m_3\chi_1\chi_2 + \frac{1}{3}\lambda(\chi_0^3 + \chi_1^3 + \chi_2^3 + 6\chi_0\chi_1\chi_2)$$

(3.21)

Ma, hep-ph/0612022

This will throw new light on the formula (1.1).
4 Summary

The charged lepton mass formula is not supernatural beings, so that the investigation should be done along the conventional mass matrix approach based on the field theory.

A promising possibility has come up: The charged lepton mass formula (1.1) and the tribimaximal mixing (2.1) are related each other and both can be understood from $S_3$ or $A_4$.

Our next step is to attack the quark masses and mixing by applying the present approach.
I am happy that the understanding of the charged lepton mass formula (1.1) is, step by step, proceeding from the mysterious world to the actual world.