Neutrino Mixing based on Mass Matrices with a $2 \leftrightarrow 3$ Symmetry

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Abstract

Under an assumption that the $2 \leftrightarrow 3$ symmetry is broken only through phases, we give a systematical investigation of possible lepton mass matrix forms without referring to explicit parameter values. Two types of the $2 \leftrightarrow 3$ symmetry are investigated: one is that the left- and right-handed fields (f_L, f_R) obey the symmetry, and another one is that only f_L obeys the symmetry. In latter case, in spite of no $2 \leftrightarrow 3$ symmetry in the Majorana mass matrix M_R for ν_R, the neutrino seesaw mass matrix still obeys the $2 \leftrightarrow 3$ symmetry. Possible phenomenology is discussed.

1 Introduction

We usually consider that the quarks and leptons should be understood by a unification theory. Then, the concept of "symmetry" will become important in the understanding of "flavor". It is well-known that the requirement of the $2 \leftrightarrow 3$ symmetry [1] for the neutrino mass matrix leads to the maximal mixing between the ν_2 and ν_3 components. The idea of the $2 \leftrightarrow 3$ symmetry is very promising for understanding the observed neutrino mixing.

When a mass matrix M satisfies the relation

$$T_{23} M T_{23}^{\dagger} = M, \tag{1.1}$$

where T_{23} is defined as

$$T_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{1.2}$$

the matrix M is called "$2 \leftrightarrow 3$ symmetric". The matrix form of M is explicitly expressed as

$$M = \begin{pmatrix} d & a & a \\ a & b & c \\ a & c & b \end{pmatrix}. \tag{1.3}$$

First, we would like to notice that the mass matrix which satisfies Eq.(1.1) is considered a consequence of the invariance of the mass matrix under the field transformation. Explicitly, for the Dirac mass matrix $\bar{f}_L M f_R$, Eq.(1.1) is derived by requiring the invariance under the transformation, $f_L \to T_{23}^{\dagger} f_L$ and $f_R \to T_{23}^{\dagger} f_R$. This is true if the neutrino mass matrix is derived by the seesaw mechanism, $M_\nu = m_D M_R^{-1} m_D^T$ because this matrix M_ν is invariant under $f_R \to T f_R$ with any T. Next, we focus on the transformation of ν_L and e_L. Since they forms a doublet of the electroweak symmetry, the transformation for them should be the same. That makes a big trouble to realize the reasonable neutrino mixing as we see in the next section.

Now we extend the $2 \leftrightarrow 3$ symmetry according to multiplets under the electroweak symmetry. In general, the transformation between (ν_L, e_L) and ν_R are different. This is true even we consider the SU(5) GUT. On the other hand, in the SO(10) GUT, (ν_L, e_L) and (ν_R, e_R) will be transformed under the same operator T_{23}. According to this classification, two types of $2 \leftrightarrow 3$ symmetry arise. The one (we call it Type I) is that both f_L and f_R obey the $2 \leftrightarrow 3$ symmetry. The form (1.3) is obtained for the Dirac mass matrices of the charged leptons and neutrinos. Since, in the neutrino sector, the Dirac neutrino mass matrix M_L and the Majorana mass matrix M_R for ν_R satisfy the relations $T_{23} M_L T_{23}^\dagger = M_L$ and $T_{23} M_R T_{23} = M_R$, we will find that the neutrino mass matrix $M_\nu = M_L M_R^{-1} (M_L)^T$ is also given by the form (1.3).

The other one (we call it Type II) is a case where only f_L obeys the $2 \leftrightarrow 3$ symmetry. Then, we find for the Dirac mass matrix M_L^f (we define a Dirac mass matrix M_L^f as $\bar{f}_L M_L^f f_R$)

$$T_{23} M_L^f = M_L^f. \tag{1.4}$$

and the explicit form of the mass matrix M_L^f is given by

$$M_L^f = \begin{pmatrix} a_1 & b_1 & c_1 \\ a & b & c \\ a & b & c \end{pmatrix}. \tag{1.5}$$

The neutrino mass matrix $M_\nu = M_L^\nu M_R^{-1} (M_L^\nu)^T$ is given as a special case of Eq.(1.3) by taking $b = c$ as we shall see later.

Note that, in the both types I and II, the Hermitian matrix defined by $H_f = M_f M_f^\dagger$ satisfies the constraint

$$T_{23} H_f T_{23}^\dagger = H_f, \tag{1.6}$$

independently whether the mass matrix has the form in Eq.(1.3) or (1.5).

Now the neutrino mixing matrix U is given by

$$U = U_{Le}^\dagger U_{L\nu}, \tag{1.7}$$

where U_{Lf} are defined by

$$U_{Lf}^\dagger H_f U_{Lf} = \mathrm{diag}(m_{f1}^2, m_{f2}^2, m_{f3}^2) \equiv D_f^2. \tag{1.8}$$

From the argument given above, we learned that as far as the mixing matrix U is concerned, the structure of the neutrino mixing matrix is independent of the mass matrices of Types I or II. Only difference arises in the mass spectrum.

The purposes of the present paper is to investigate the general properties of the models with the $2 \leftrightarrow 3$ symmetry, paying attention to the difference between types I and II, and taking relations to the grand unification (GUT) scenarios into consideration. Although we investigate the masses and mixings in the lepton sectors, the formulation in this paper is also applicable to the quark sectors. Since, in the quark sectors, there is essentially no complexity about the mass spectrum such as the inverse hierarchy as in the neutrino sector, the application is more straightforward. Therefore, we will investigate only the lepton sectors in this paper.

2 Extended $2 \leftrightarrow 3$ symmetry and the neutrino mixing

In this section, we will demonstrate that the $2 \leftrightarrow 3$ symmetry in the exact meaning cannot explain the observed neutrino mixing. For the convenience of the discussion in later, let us introduce the so-called extended $2 \leftrightarrow 3$ operator $T_{23}(2\delta)$ [2]

$$T_{23}(2\delta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & e^{i2\delta} \\ 0 & e^{-i2\delta} & 0 \end{pmatrix}, \tag{2.1}$$

instead of the operator (1.2). The operator $T_{23}(2\delta)$ is unitary and Hermitian. We obtain the constraint

$$T_{23}(2\delta)MM^\dagger T_{23}^\dagger(2\delta) = MM^\dagger, \tag{2.2}$$

for the Hermitian matrix $MM^\dagger$ irrespective of Type I or II. Note that we can express the operator (2.1) as

$$T_{23}(2\delta) = P_{23}(2\delta)T_{23} = P_{23}(\delta)T_{23}P_{23}^\dagger(\delta) = T_{23}P_{23}^\dagger(2\delta), \tag{2.3}$$

where $T_{23} = T_{23}(0)$ and

$$P_{23}(\delta) = \text{diag}(1, e^{i\delta}, e^{-i\delta}). \tag{2.4}$$

Therefore, we can express the constraint (2.2) as

$$MM^\dagger = P_{23}(\delta)T_{23}P_{23}^\dagger(\delta)MM^\dagger P_{23}(\delta)T_{23}P_{23}^\dagger(\delta). \tag{2.5}$$

Now we define

$$H = P_{23}^\dagger(\delta)MM^\dagger P_{23}(\delta), \tag{2.6}$$

then we find

$$H = T_{23}HT_{23}, \tag{2.7}$$

where H is a Hermitian matrix

In general, the Hermitian matrix H which satisfies the constraint (2.6) can be expressed by the form

$$H = \begin{pmatrix} D & Ae^{i\phi} & Ae^{i\phi} \\ Ae^{-i\phi} & B & C \\ Ae^{-i\phi} & C & B \end{pmatrix}, \tag{2.8}$$

where A, B, C and D are real, so that H can be transformed to a real matrix $\widetilde{H}$ as

$$P_1^\dagger(\phi)HP_1(\phi) = \widetilde{H}, \tag{2.9}$$

where

$$P_1(\phi) = \mathrm{diag}(e^{i\phi}, 1, 1). \tag{2.10}$$

It is also well-known that the $2 \leftrightarrow 3$ symmetric real matrix $\widetilde{H}$ is diagonalized by a rotation $R(\theta)$ as

$$R^T(\theta)\widetilde{H}R(\theta) = \widetilde{H}_D \equiv \mathrm{diag}(m_1^2, m_2^2, m_3^2), \tag{2.11}$$

where

$$R(\theta) = \begin{pmatrix} c_\theta & s_\theta & 0 \\ -\frac{1}{\sqrt{2}}s_\theta & \frac{1}{\sqrt{2}}c_\theta & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}}s_\theta & \frac{1}{\sqrt{2}}c_\theta & \frac{1}{\sqrt{2}} \end{pmatrix}, \tag{2.12}$$

$$s_\theta \equiv \sin\theta = \sqrt{\frac{D - m_1^2}{m_2^2 - m_1^2}}, \quad c_\theta \equiv \cos\theta = \sqrt{\frac{m_2^2 - D}{m_2^2 - m_1^2}}, \tag{2.13}$$

$$\begin{aligned} m_1^2 &= \tfrac{1}{2}\left(B + C + D - \sqrt{8A^2 + (B + C - D)^2}\right), \\ m_2^2 &= \tfrac{1}{2}\left(B + C + D + \sqrt{8A^2 + (B + C - D)^2}\right), \\ m_3^2 &= B - C. \end{aligned} \tag{2.14}$$

As a result, the Hermitian matrix $MM^\dagger$ is diagonalized by

$$U = P_{23}(\delta)P_1(\phi)R(\theta), \tag{2.15}$$

as

$$U^\dagger MM^\dagger U = \widetilde{H}_D. \tag{2.16}$$

Although we did not consider the size of masses, the ordering of them is needed. Therefore, the unitary matrix to diagonalize the mass matrix in an proper mass ordering is given by UT, where T is the matrix to exchange the mass ordering. Then, we find the neutrino mixing matrix defined by (1.7) as

$$U = U_e^\dagger U_\nu = T_e^T R^T(\theta_e)P_e^\dagger P_\nu R(\theta_\nu)T_\nu, \tag{2.17}$$

where

$$P_f = P_{23}(\delta_f)P_1(\phi_f) = \mathrm{diag}(e^{i\phi_f}, e^{i\delta_f}, e^{-i\delta_f}). \tag{2.18}$$

Here, we recall that the operation (2.1) must be the same for ν_L and e_L, so that, in the expression, δ_e is exactly equal to δ_ν. Therefore, we obtain

$$U = T_e^T U_0 T_\nu \equiv T_e^T \begin{pmatrix} s_e s_\nu + c_e c_\nu e^{i\phi} & -s_e c_\nu + c_e s_\nu e^{i\phi} & 0 \\ -c_e s_\nu + s_e c_\nu e^{i\phi} & c_e c_\nu + s_e s_\nu e^{i\phi} & 0 \\ 0 & 0 & 1 \end{pmatrix} T_\nu, \tag{2.19}$$

where $\phi = \phi_\nu - \phi_e$. Obviously, the mixing matrix (2.19) cannot give the observed values [3, 4] $\tan^2\theta_{12} \simeq 1/2$ and $\sin^2 2\theta_{23} \simeq 1$ simultaneously. (It is a general feature for any flavor symmetry with a transformation $f_L \to U_X f_L$ that we obtain only a family-mixing between two families. See Ref.[5].)

3 Extended $2 \leftrightarrow 3$ symmetry with the breaking term

We saw in the previous section that the $2 \leftrightarrow 3$ symmetry which arises as a consequence of the transformation for fields cannot reproduce the observed neutrino mixing. However, we consider that the $2 \leftrightarrow 3$ symmetry is still useful from the phenomenological point of view. Therefore, from the phenomenological point of view, we assume [6] that the $2 \leftrightarrow 3$ symmetry is broken only through the phase parameters. Hereafter, we will use the extended $2 \leftrightarrow 3$ symmetry operator (2.1) in the phenomenological meaning, and we will consider the case $\delta_e \neq \delta_\nu$ in the left-handed sectors.

3.1) Charged lepton mass spectrum

First, we investigate the $2 \leftrightarrow 3$ symmetry of Type II. The mass matrix M_L^e for the charged leptons must also satisfy the relation

$$T_{23}(2\delta) M_L^e = M_L^e, \tag{3.1}$$

where, for convenience, we have dropped the index "e" from δ_e. Then, the explicit form of M_L^e is also given by

$$M_L^e = \begin{pmatrix} a_1 & b_1 & c_1 \\ ae^{i\delta} & be^{i\delta} & ce^{i\delta} \\ ae^{-i\delta} & be^{-i\delta} & ce^{-i\delta} \end{pmatrix} = P_{23}(\delta) \begin{pmatrix} a_1 & b_1 & c_1 \\ a & b & c \\ a & b & c \end{pmatrix}, \tag{3.2}$$

where the parameters $a, b, \cdots$ in M_L^e can be complex. Therefore, we obtain the Hermitian matrix

$$M_L^e (M_L^e)^\dagger = P_{23}(\delta) \begin{pmatrix} D & Ae^{i\phi} & Ae^{i\phi} \\ Ae^{-i\phi} & B & B \\ Ae^{-i\phi} & B & B \end{pmatrix} P_{23}^\dagger(\delta), \tag{3.3}$$

where

$$\begin{aligned} A &= |aa_1^* + bb_1^* + cc_1^*|, \\ B &= |a|^2 + |b|^2 + |c|^2, \\ D &= |a_1|^2 + |b_1|^2 + |c_1|^2. \end{aligned} \tag{3.4}$$

Then, we can obtain a real matrix $\widetilde{H}_e$ as

$$\widetilde{H}_e = P_1^\dagger(\phi) P_{23}^\dagger(\delta) M_L^e (M_L^e)^\dagger P_{23}(\delta) P_1(\phi). \tag{3.5}$$

From the formula (2.14), we obtain

$$m_{e3} = 0, \tag{3.6}$$

because of $B = C$ in this case. Therefore, Type II transformation in charged lepton sector cannot give a realistic mass spectrum.

Next, we investigate the case of Type I, i.e.

$$\ell_L \to T_{23}(2\delta_L)\ell_L, \quad e_R \to T_{23}(2\delta_R)e_R. \tag{3.7}$$

The case (3.7) may be realized in an SU(5)-GUT model[1]. In this case, instead of the constraint (3.1), we have the constraint

$$T_{23}(2\delta_L)M_L^e T_{23}^\dagger(2\delta_R) = M_L^e. \tag{3.8}$$

The explicit form of M_L^e is given by

$$M_L^e = \begin{pmatrix} d & a'e^{-i\delta_R} & a'e^{i\delta_R} \\ ae^{i\delta_L} & be^{i(\delta_L-\delta_R)} & ce^{-i(\delta_L+\delta_R)} \\ ae^{-i\delta_L} & ce^{i(\delta_L+\delta_R)} & be^{-i(\delta_L-\delta_R)} \end{pmatrix} = P_{23}(\delta_L) \begin{pmatrix} d & a' & a' \\ a & b & c \\ a & c & b \end{pmatrix} P_{23}^\dagger(\delta_R), \tag{3.9}$$

so that we obtain

$$M_L^e(M_L^e)^\dagger = P_{23}(\delta_L) \begin{pmatrix} D & Ae^{i\phi} & Ae^{i\phi} \\ Ae^{-i\phi} & B & C \\ Ae^{-i\phi} & C & B \end{pmatrix} P_{23}^\dagger(\delta_L), \tag{3.10}$$

where

$$\begin{aligned} A &= |ad^* + (b+c)a'^*|, \\ B &= |a|^2 + |b|^2 + |c|^2, \\ C &= |a|^2 + 2|b||c|\cos(\beta-\gamma), \\ D &= |d|^2 + 2|a'|^2, \end{aligned} \tag{3.11}$$

where β and γ are defined by $b = |b|e^{i\beta}$ and $c = |c|e^{i\gamma}$, respectively. Therefore, since

$$m_{e3}^2 = B - C = |b|^2 + |c|^2 - 2|b||c|\cos(\beta-\gamma) = |b-c|^2, \tag{3.12}$$

we can obtain $m_{e3} \neq 0$ when $b \neq c$.

In both cases, Types I and II, the Hermitian matrix $M_L^e(M_L^e)^\dagger$ is diagonalized by a unitary matrix

$$U_e = P_{23}(\delta_e)P_1(\phi_e)R(\theta_e), \tag{3.13}$$

as

$$U_e^\dagger M_L^e(M_L^e)^\dagger U_e = D_e^2 \equiv (m_{e1}^2, m_{e2}^2, m_{e3}^2). \tag{3.14}$$

3.2) Neutrino mass spectrum

[1]In SU(5) GUT models, the matter fields (ν_L, e_L) and ν_R^c are assigned to $\bar{5}_L$ and 10_L of SU(5), respectively. Therefore, in general, flavor symmetries for $\bar{5}_L$ and 10_L can be different from each other. However, in this paper, we consider a specific case in which $\bar{5}_L$ and 10_L obey the same flavor symmetry. Hereafter, we call it an SU(5) $\bar{5}_L + 10_L$ model for convenience.

We consider that the neutrino masses are generated by the seesaw mechanism

$$M_\nu = M_L^\nu M_R^{-1} (M_L^\nu)^T, \tag{3.15}$$

where M_L^ν and M_R are defined by $\bar{\nu}_L M_L^\nu \nu_R$ and $\bar{\nu}_R^c M_R \nu_R$ ($\nu_R^c \equiv C\bar{\nu}_R^T$), respectively. The Dirac mass matrix M_L^ν is given by the form similar to (3.9) or (3.2) according as Type-I or Type-II. In Type-II, we obtain the neutrino mass matrix form

$$M_\nu = P_{23}(\delta) \begin{pmatrix} D & A & A \\ A & B & B \\ A & B & B \end{pmatrix} P_{23}(\delta), \tag{3.16}$$

where

$$\begin{aligned}
A &= aa_1 d_R^{-1} + bb_1 b_R^{-1} cc_1 b_R'^{-1} + (ab_1 + a_1 b) a_R^{-1} + (ac_1 + a_1 c) a_R'^{-1} + (b_1 bc_1 + bc_1) c_R, \\
B &= b^2 b_R^{-1} + c^2 b_R'^{-1} + a^2 d_R + 2bc c_R^{-1} + 2ab a_R^{-1} + 2ac a_R'^{-1}, \\
D &= b_1^2 b_R^{-1} + c_1^2 b_R'^{-1} + a_1^2 d_R + 2b_1 c_1 c_R^{-1} + 2a_1 b_1 a_R^{-1} + 2a_1 c_2 a_R'^{-1},
\end{aligned} \tag{3.17}$$

$$M_R^{-1} = \begin{pmatrix} d_R^{-1} & a_R^{-1} & a_R'^{-1} \\ a_R^{-1} & b_R^{-1} & c_R^{-1} \\ a_R'^{-1} & c_R^{-1} & b_R'^{-1} \end{pmatrix}. \tag{3.18}$$

Since the neutrino masses $m_{\nu i}$ in Type-II are given by

$$\begin{aligned}
m_{\nu 1} &= \tfrac{1}{2}\left(B + C + D - \sqrt{8A^2 + (B + C - D)^2}\right), \\
m_{\nu 2} &= \tfrac{1}{2}\left(B + C + D + \sqrt{8A^2 + (B + C - D)^2}\right), \\
m_{\nu 3} &= B - C,
\end{aligned} \tag{3.19}$$

with $C = B$, we obtain

$$m_{\nu 3} = 0. \tag{3.20}$$

On the other, in Type I, such the constraint (3.20) does not appear.

In both cases, Types I and II, the Hermitian matrix $M_\nu M_\nu^\dagger$ is diagonalized by a unitary matrix

$$U_\nu = P_{23}(\delta_\nu) P_1(\phi_\nu) R(\theta_\nu), \tag{3.21}$$

as

$$U_\nu^\dagger M_\nu M_\nu^\dagger U_\nu = D_\nu^2 \equiv (m_{\nu 1}^2, m_{\nu 2}^2, m_{\nu 3}^2), \tag{3.22}$$

where $R(\theta_\nu)$ is defined by Eq.(2.12) with

$$s_\nu \equiv \sin\theta_\nu = \sqrt{\frac{D - m_{\nu 1}}{m_{\nu 2} - m_{\nu 1}}}, \quad c_\nu \equiv \cos\theta_\nu = \sqrt{\frac{m_{\nu 2} - D}{m_{\nu 2} - m_{\nu 1}}}. \tag{3.23}$$

3.3) Neutrino mixing matrix

So far, we have used the notation (f_1, f_2, f_3) for the mass eigenstates of the fundamental fermions f, whose masses m_{fi} have been defined by Eq.(2.14). Hereafter, in order to distinguish the mass-eigenstates (e, μ, τ) and (ν_1, ν_2, ν_3) in the conventional notations from the mass-eigenstates whose masses m_i are defined by Eq.(2.14), we denote the states whose masses are defined by Eq.(2.14) as f_i^0. The states (ν_1, ν_2, ν_3) and $(\nu_e, \nu_\mu, \nu_\tau)$, which is the $\mathrm{SU(2)}_L$ partner of the charged lepton state (e, μ, τ), are related by

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = U \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}, \tag{3.24}$$

with the neutrino mixing matrix U in the conventional notation. Here, the neutrino mixing matrix U in Eq.(3.24) is given by

$$U = U_e^\dagger U_\nu. \tag{3.25}$$

On the other hand, as seen in Secs.2 and 3, the mass matrices $M_\nu M_\nu^\dagger$ and $M_L^e (M_L^e)^\dagger$ are diagonalized by unitary matrices (3.21) and (3.13) (we denote them $U_{0\nu}$ and U_{0e}), respectively. When we define the mixing matrix

$$U_0 = U_{0e}^\dagger U_{0\nu} = R^T(\theta_e) P R(\theta_\nu), \tag{3.26}$$

where

$$P = \mathrm{diag}(e^{i\phi}, e^{i\delta}, e^{-i\delta}), \tag{3.27}$$

$\phi = \phi_\nu - \phi_e$ and $\delta = \delta_\nu - \delta_e$. the mixing matrix U_0 does not always denote the observed neutrino mixing matrix U. When we define the observed fermions (e, μ, τ) and (ν_1, ν_2, ν_3) as

$$\begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = T_{ijk} \begin{pmatrix} \nu_1^0 \\ \nu_2^0 \\ \nu_3^0 \end{pmatrix}, \quad \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix} = T_{lmn} \begin{pmatrix} e_1^0 \\ e_2^0 \\ e_3^0 \end{pmatrix}, \tag{3.28}$$

the observed neutrino mixing matrix U is given by

$$U = T_{lmn} U_0 T_{ijk}^T, \tag{3.29}$$

where T_{ijk} denotes the exchange operator $(f_1^0, f_2^0, f_3^0) \to (f_i^0, f_j^0, f_k^0)$. However, as we discuss below, the possible choices of T_{ijk} are not so many.

The explicit form of the matrix U_0 is given by

$$U_0 = \begin{pmatrix} c_e c_\nu e^{i\phi} + s_e s_\nu \cos\delta & c_e s_\nu e^{i\phi} - s_e c_\nu \cos\delta & i s_e \sin\delta \\ s_e c_\nu e^{i\phi} - c_e s_\nu \cos\delta & s_e s_\nu e^{i\phi} + c_e c_\nu \cos\delta & -i c_e \sin\delta \\ i s_\nu \sin\delta & -i c_\nu \sin\delta & \cos\delta \end{pmatrix}. \tag{3.30}$$

Table 1: Possible constraints on the Dirac mass matrices m_L^f: Models A, B, C, and D are defined according as the constraint types.

Type	Type II for M_L^ν	Type I for M_L^ν
Type II for M_L^e	Model A: non-GUT type $m_{e3} = m_{\nu 3} = 0$	Model D: unrealistic $m_{e3} = 0$ & $m_{\nu 3} \neq 0$
Type I for M_L^e	Model B: SU(5) $\bar{5}_L + 10_L$ model $m_{e3} \neq 0$ & $m_{\nu 3} = 0$	Model C: SO(10)-GUT type $m_{e3} \neq 0$ & $m_{\nu 3} \neq 0$

Obviously, the cases (3.29) with $\delta = 0$ are ruled out as we have already discussed in Sec.2.

For convenient, we name Models A, B, C and D for combinations of Types I and II for M_L^e and M_L^ν as shown in Table 1. In Model A, since only the left-handed fields f_L obey the $2 \leftrightarrow 3$ symmetry, the model cannot be embedded into a GUT scenario. In Model B, the fields $\ell_L = (\nu_L, e_L)$ and e_R obey the $2 \leftrightarrow 3$ symmetry, but the field ν_R is free from the symmetry, so that the model can be embedded into SU(5) GUT [see a footnote below Eq.(3.71)]. In Model C, all fields $\ell_L = (\nu_L, e_L)$, e_R and ν_R obey the $2 \leftrightarrow 3$ symmetry, so that the model can be embedded into SO(10) GUT. Model D is unlikely, so that we will not investigate this case.

In Models A and D with Type-II symmetry in the charged lepton sector, we obtain $m_{e3} = 0$, so that the cases are ruled out.

In Model B (an SU(5) $\bar{5}_L + 10_L$ model), we can obtain $m_{e3} \simeq 0$ (but $m_{e3} \neq 0$) because of $b \simeq c$. (In Model B, although we can, in principle, consider any value of m_{e3}, we have assumed $b \simeq c$ because the case $b \simeq c$ can reasonably be realized in most practical models with $2 \leftrightarrow 3$ symmetry.) Therefore, we may suppose a case $m_{e2}^2 > m_{e1}^2 > m_{e3}^2$ in the model. Such the case means the assignment

$$\begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix}_L = \begin{pmatrix} e_3^0 \\ e_1^0 \\ e_2^0 \end{pmatrix}_L = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} e_1^0 \\ e_2^0 \\ e_3^0 \end{pmatrix}_L \equiv T_{312} \begin{pmatrix} e_1^0 \\ e_2^0 \\ e_3^0 \end{pmatrix}_L . \tag{3.31}$$

Then, from the relation $U_e = U_{e0} T_{312}^T$, the observed neutrino mixing matrix U is described by

$$U = T_{312} U_0 = \begin{pmatrix} i s_\nu \sin\delta & -i c_\nu \sin\delta & \cos\delta \\ c_e c_\nu e^{i\phi} + s_e s_\nu \cos\delta & c_e s_\nu e^{i\phi} - s_e c_\nu \cos\delta & i s_e \sin\delta \\ s_e c_\nu e^{i\phi} - c_e s_\nu \cos\delta & s_e s_\nu e^{i\phi} + c_e c_\nu \cos\delta & -i c_e \sin\delta \end{pmatrix} , \tag{3.32}$$

if we regard the observed neutrino states (ν_1, ν_2, ν_3) as $(\nu_1^0, \nu_2^0, \nu_3^0)$ with $m_{\nu_3} = 0$, whose case corresponds to the inverse hierarchy. (Such an inverted assignment between up- and down-sectors was first proposed by Matsuda and Nishiura [7].) The case (3.32) predicts

$$\tan^2\theta_{solar} = \frac{|U_{12}|^2}{|U_{11}|^2} = \frac{c_\nu^2}{s_\nu^2} = \frac{m_{\nu 2} - D_\nu}{D_\nu - m_{\nu 1}}, \tag{3.33}$$

$$\sin^2 2\theta_{atm} = 4|U_{23}|^2|U_{33}|^2 = \sin^2 2\theta_e \sin^4\delta = \sin^2 2\theta_e(1-|U_{13}|^2)^2, \tag{3.34}$$

where s_e and c_e are given by Eq.(2.13). In order to give $|U_{13}|^2 \simeq 0$, the condition $\cos\delta \simeq 0$ ($\delta \simeq \pi/2$) is required. In order to $\sin^2 2\theta_e = 1$ ($s_e^2 = c_e^2 = 1/2$), the relation $2D_e = m_{e1}^2 + m_{e2}^2$ (i.e. $D_e = B_e + C_e$) is required from Eq.(2.14). Then, the masses (2.14) are given by

$$\begin{array}{l} m_{e3}^2 = B_e - C_e = |b_e - c_e|^2, \\ m_{e1}^2 = D_e - \sqrt{2}A_e, \\ m_{e2}^2 = D_e + \sqrt{2}A_e. \end{array} \tag{3.35}$$

Therefore, a suitable choice of the parameter values of M_L^e can give $\sin^2 2\theta_e = 1$ keeping $m_{e2}^2 > m_{e1}^2 > m_{e3}^2$. Also, a suitable choice of the parameter values of M_ν can give a reasonable value of (3.33). If these conditions are satisfied, the model B is preferable. However, note that the parameter value $\delta \simeq \pi/2$ cannot be realized unless SU(2)$_L$ is broken.

By the way, the case $m_{\nu 3} = 0$ does not always mean the inverse hierarchy of neutrino masses. At present, as far as the observed neutrino masses m_{ν_i} satisfy the relation $(m_{\nu 2}^2 - m_{\nu 1}^2)/|(m_{\nu 3}^2 - m_{\nu 2}^2)| \sim 10^{-2}$, we may consider any cases $U = T_{312}U_0T_{ijk}^T$. Therefore, even the case $m_{\nu 3} = 0$, we can consider a case of the normal hierarchy: $(\nu_1, \nu_2, \nu_3) = (\nu_3^0, \nu_1^0, \nu_2^0)$. Then, in Model B with $c_e \simeq b_e$, the neutrino mixing matrix U is given by

$$U = T_{312}U_0T_{312}^T = \begin{pmatrix} \cos\delta & is_\nu \sin\delta & -ic_\nu \sin\delta \\ is_e \sin\delta & c_e c_\nu e^{i\phi} + s_e s_\nu \cos\delta & c_e s_\nu e^{i\phi} - s_e c_\nu \cos\delta \\ -ic_e \sin\delta & s_e c_\nu e^{i\phi} - c_e s_\nu \cos\delta & s_e s_\nu e^{i\phi} + c_e c_\nu \cos\delta \end{pmatrix}. \tag{3.36}$$

In order to give $\tan^2\theta_{solar} \simeq 1/2$ and $\sin^2 2\theta_{atm} \simeq 1$, we have to consider $c_\nu \simeq 0$. From the expression (3.23), the limit of $c_\nu = 0$ requires $m_{\nu 2} = D_\nu$, which leads $A_\nu = 0$ and gives the mass spectrum $m_{\nu 1} = D_\nu$, $m_{\nu 2} = 2B_\nu$ and $m_{\nu 3} = 0$. If we choose $B_\nu^2 \gg D_\nu^2$ in the neutrino sector, we can give a reasonable value of $R = \Delta m_{solar}^2/\Delta m_{atm}^2$ because of $R = (m_1^2 - m_3^2)/(m_2^2 - m_1^2) = D_\nu^2/(4B_\nu^2 - D_\nu^2)$ in the normal mass hierarchy. Therefore, we cannot rule out this case (Model B with $m_{e2}^2 \gg m_{e1}^2 \gg m_{e3}^2$ and $m_{\nu 2}^2 \gg m_{\nu 1}^2 \gg m_{\nu 3}^2$ in a normal hierarchy). However, we must accept a phenomenological value $\tan^2\delta \simeq 1/2$ ($\delta \simeq 35.3^\circ$) in order to understand $\tan^2\theta_{solar} \simeq 1/2$.

So far, we have consider the case with $c_e \simeq b_e$ (i.e. $m_{e3}^2 \ll m_{e1}^2 \ll m_{e2}^2$) for the charged lepton masses in Model B. We can also consider the case $m_{e1}^2 \ll m_{e2}^2 \ll m_{e3}^2$ in Model B. In Model B, the neutrino masses are still given by $m_{\nu 3}^2 = 0 < m_{\nu 1}^2 < m_{\nu 2}^2$, so that the cases $U = T_{123}U_0T_{312}^T$ and $U = T_{123}U_0T_{123}^T$ correspond to the normal and inverse hierarchies, respectively. The explicit form of U for the case $U = T_{123}U_0T_{123}^T$ has been given in (3.30) because $U = T_{123}U_0T_{123}^T = U_0$. The explicit form of the case $U = T_{123}U_0T_{312}^T$ is given by

$$U = T_{123}U_0T_{312}^T = \begin{pmatrix} is_e \sin\delta & c_e c_\nu e^{i\phi} + s_e s_\nu \cos\delta & c_e s_\nu e^{i\phi} - s_e c_\nu \cos\delta \\ -ic_e \sin\delta & s_e c_\nu e^{i\phi} - c_e s_\nu \cos\delta & s_e s_\nu e^{i\phi} + c_e c_\nu \cos\delta \\ \cos\delta & is_\nu \sin\delta & -ic_\nu \sin\delta \end{pmatrix}. \tag{3.37}$$

Table 2: Possible neutrino mixing matrix form in Model B.

$m_{\nu 0i}$	$m^2_{\nu 03} = 0 < m^2_{\nu 01} < m^2_{\nu 02}$			
m_{e0i}	$m^2_{e03} < m^2_{e01} < m^2_{e02}$		$m^2_{e01} < m^2_{e02} < m^2_{e03}$	
Hierarchy	Normal	Inverse	Normal	Inverse
U	$T_{312}U_0T^T_{312}$	$T_{312}U_0T^T_{123}$	$T_{123}U_0T^T_{312}$	$T_{123}U_0T^T_{123}$
Limit of	$\tan^2\delta = 1/2$	$\delta = \pi/2$	$\tan^2\delta = 5$	$\delta = \pi/4$
$\sin^2 2\theta_{23} = 1$	$s^2_e = 1/2$	$s^2_e = 1/2$	$s^2_e = 4/5$	$s^2_e = 0$
& $\tan^2\theta_{12} = 1/2$	$s^2_\nu = 1$	$s^2_\nu = 2/3$	$s^2_\nu = 2/5$	$s^2_\nu = 1/3$

In order to see whether those cases cannot be ruled out or not, it is convenient to see whether we can take or not possible parameter values in the limit of $\tan^2\theta_{solar} = 1/2$, $\sin^2 2\theta_{atm} = 1$ and $|U_{13}|^2 = 0$, without contradicting with the observed neutrino mass hierarchy. The results are listed in Table 2. All cases are acceptable if we neglect the problem whether such a set of the parameter values is natural or not, although we think that the case with $U = T_{123}U_0T^T_{312}$ is unlikely.

In Model C, since we can take any order of m^2_i, we cannot say any definite conclusion (predictions) without giving the explicit mass matrix parameters. Therefore, for the case C, we do not give a table such as Table 2.

4 Summary

In conclusion, we have systematically investigated possible lepton mass matrix forms and mixings under the expended $2 \leftrightarrow 3$ symmetry.

As we discussed in Sec.2, if we require an exact (unbroken) $2 \leftrightarrow 3$ symmetry, which corresponds to the case $\delta \equiv \delta_\nu - \delta_e = 0$, we cannot obtain realistic neutrino mixing matrix. Therefore, from the phenomenological point of view, we have assumed that the $2 \leftrightarrow 3$ symmetry is broken only through the phase parameters, i.e. by the parameter $\delta \neq 0$. In this paper, we did not argue the origin of $\delta \neq 0$. We consider that the phase difference value $\delta = \delta_\nu - \delta_e$ is not always small. Rather, we consider that it is likely that the value δ takes a specific (not always small) value as $\delta = \pi/2$, $\delta = \pi/4$, and so on.

We have investigated two types of the $2 \leftrightarrow 3$ symmetry: one (Type I) is that the left- and right-handed fields (f_L, f_R) obey the symmetry, and another one (Type II) is that only f_L obeys the symmetry. Note that even in Type II, in spite of no $2 \leftrightarrow 3$ symmetry in the Majorana mass matrix M_R for ν_R, the neutrino seesaw mass matrix still obey the $2 \leftrightarrow 3$ symmetry. However, we have concluded that the fermion mass m_3 is always zero in Type II. Therefore, the possibility that the charged lepton sector obeys the $2 \leftrightarrow 3$ symmetry of Type II is ruled out. We have been interested in the case B classified in Table 1, where the neutrino sector obeys the $2 \leftrightarrow 3$ symmetry of Type II, because we consider a model with an SU(5)-GUT type scenario [8]. In this case, we have only four cases of the neutrino mixing matrix. The results are summarized in Table 2.

We are also interested in a model with an SO(10)-type scenario. In this case (Model C), the right-handed neutrino ν_R is also transformed as $\nu_R \to T_{23}\nu_R$, so that we can consider any value of $m_{\nu_{03}} \neq 0$ and any mixing matrix form (3.29). However, in the SO(10)-GUT model, a more strict constraint on the neutrino mass matrix appears because the neutrino mass matrix form is strictly related to the quark and charged lepton mass matrices, so that most naive SO(10) models have, at present, not succeeded [9] in giving reasonable fits for all the masses and mixings in the quark and lepton sectors, even without the $2 \leftrightarrow 3$ symmetry.

In the practical point of view, we think that there is a possibility to build a realistic model (SU(5) $\bar{5}_L + 10_L$ model) based on SU(5)-GUT rather than SO(10). In Model B, we are interested in the case of an inverse neutrino mass hierarchy, because the case $\delta = \pi/2$ is likely. The case predicts the effective electron neutrino mass $\langle m_{\nu e} \rangle$ is of the order of $\sqrt{\Delta m^2_{atm}} \simeq 0.05$ eV, which is within the reach of the next generation experiments of the neutrinoless double beta decay.

We hope that the present investigation will be helpful to investigate more explicit model based on a GUT scenario.

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