Broken SU(3) Flavor Symmetry
and Tribimaximal Neutrino Mixing

Yoshio Koide
IHERP, Osaka University,
1-16 Machikaneyama, Toyonaka, Osaka 560-0043, Japan
E-mail address: koide@het.phys.sci.osaka-u.ac.jp

Abstract

Recent work on a lepton mass matrix model based on an SU(3) flavor symmetry which
is broken into $S_4$ is reviewed. The flavor structures of the masses and mixing are caused
by VEVs of SU(2)$_L$-singlet scalars $\phi$ which are nonets ($8+1$) of the SU(3) flavor symmetry,
and which are broken into $2+3+3'$ and $1$ of $S_4$. If we require the invariance under
the transformation $(\phi^{(8)}, \phi^{(1)}) \rightarrow (-\phi^{(8)}, +\phi^{(1)})$ for the superpotential of the nonet field
$\phi^{(8+1)}$, the model leads to a beautiful relation for the charged lepton masses. The observed
tribimaximal neutrino mixing is understood by assuming two SU(3) singlet right-handed
neutrinos $\nu_R^{(\pm)}$ and an SU(3) triplet scalar $\chi$.

1 Introduction

The observed mass spectra and mixings of the fundamental particles will provide promising
clues to unified understanding of the quarks and leptons, especially, to the understanding of the
"flavor". In the present paper, we notice the following observed characteristic features in the
lepton sector [1]:
(i) The observed charged lepton masses ($m_e, m_\mu, m_\tau$) satisfy the relation [2, 3]

$$m_e + m_\mu + m_\tau = \frac{2}{3}(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2,$$

(1.1)

with remarkable precision;
(ii) The observed neutrino mixing $U_\nu$ is approximately given by the so-called tribimaximal
mixing [4]

$$U_{TB} = \begin{pmatrix}
\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}
\end{pmatrix},$$

(1.2)

which suggests that the mixing can be described by Clebsh-Gordan-like coefficients. Therefore,
for a start, in the present paper, we investigate the lepton masses and mixings. The purpose of
the present paper is to review a recent attempt [5] to investigate the mass relation (1.1) and the
tribimaximal mixing.

In order to understand the relation (1.1), for example, we assume that there are three
scalars $\phi_i$ ($i = 1, 2, 3$), and the values of the charged lepton masses $m_{ei}$ are proportional to the

\footnote{Contributed paper to XXIII International Symposium on Lepton and Photon Interactions at High Energy (Lepton-Photon 2007), Aug 13-18, 2007, Daegu, Korea.}
square of the vacuum expectation values (VEVs) \( v_i = \langle \phi_i \rangle \) of the scalars \( \phi_i \), \( m_{ei} = kv_i^2 \) (in the Ref.[3, 6, 7], for instance, a seesaw type model \((M_e)_{ij} = \delta_{ij}v_i(M_E)^{-1}v_j \) has been assumed). We define singlet \( \phi_\sigma \) and doublet \((\phi_\pi, \phi_\eta)\) of a permutation symmetry \( S_3 \) [8] by

\[
\begin{pmatrix}
\phi_\pi \\
\phi_\eta \\
\phi_\sigma
\end{pmatrix} = \begin{pmatrix}
0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}
\end{pmatrix}
\begin{pmatrix}
\phi_1 \\
\phi_2 \\
\phi_3
\end{pmatrix},
\]

from the three objects \((\phi_1, \phi_2, \phi_3)\), and we consider the following \( S_3 \) invariant scalar potential \( V(\phi) \) [3, 9, 10]:

\[
V(\phi) = m^2(\phi_\pi^2 + \phi_\eta^2 + \phi_\sigma^2) + \lambda_1(\phi_\pi^2 + \phi_\eta^2 + \phi_\sigma^2)^2 + \lambda_2\phi_\sigma^2(\phi_\pi^2 + \phi_\eta^2).
\]

(1.4)

The minimizing condition of the potential (1.4) leads to the relation

\[
v_\pi^2 + v_\eta^2 = v_\sigma^2.
\]

(1.5)

The relation (1.5) means

\[
v_1^2 + v_2^2 + v_3^2 = \frac{2}{3}(v_1 + v_2 + v_3)^2,
\]

(1.6)

because

\[
v_1^2 + v_2^2 + v_3^2 = v_\pi^2 + v_\eta^2 + v_\sigma^2 = 2v_\sigma^2 = 2\left(\frac{v_1 + v_2 + v_3}{\sqrt{3}}\right)^2.
\]

(1.7)

Therefore, we can obtain the mass relation (1.1) from the assumption \( m_{ei} \propto v_i^2 \). Here, note that although the scalar potential (1.4) is invariant under the \( S_3 \) symmetry, but it is not a general one of the \( S_3 \) invariant form. As pointed out in Ref. [10], the scalar potential with a general form cannot lead to the relation (1.5). For the derivation of the VEV relation (1.5), it is essential to choose the specific form (1.4) of the \( S_3 \) invariant terms. Similar formulation is also possible for other discrete symmetries \( A_4 \) [11] and \( S_4 \) (see below). However, in such a symmetry, we still need an additional specific selection rule. What is the meaning of such a specific selection? In the present paper, we investigate this problem by assuming that the \( S_4 \) flavor symmetry is embedded into \( SU(3) \).

On the other hand, the observed tribimaximal mixing suggests the following scenario: From the definition (1.2), we can denote the fields \((\psi_1, \psi_2, \psi_3)\) as

\[
\begin{pmatrix}
\psi_1 \\
\psi_2 \\
\psi_3
\end{pmatrix} = U_{TB} \begin{pmatrix}
\psi_\eta \\
\psi_\pi
\end{pmatrix}.
\]

(1.8)

The observed neutrino mixing (1.2) means that when the mass eigenstates of the charged leptons are given by the \((\psi_1, \psi_2, \psi_3)\) basis, the mass eigenstates of the neutrinos are given by the 

2
(ψ_η, ψ_σ, ψ_π) basis. Therefore, the problem is to find a model where the charged lepton mass eigenstates are (e_1, e_2, e_3), while the neutrino mass eigenstates are given by (ν_η, ν_σ, ν_π) with the mass hierarchy $m_\eta^2 < m_\sigma^2 \ll m_\pi^2$ (or $m_\pi^2 \ll m_\eta^2 < m_\sigma^2$). In the S_4, we can define the same relation as (1.8).

Thus, the characteristic features (1.1) and (1.2) in the lepton sector may be understood from the language of S_4 (also S_3 or A_4). However, as seen from the above review, the characteristic features (1.1) and (1.2) cannot be understood from the S_4 symmetry only. We need some additional assumptions. In the present model, we will investigate these problems under an assumption that the present S_4 symmetry is embedded into an SU(3) symmetry [12]. In the next section, the singlet $\phi_\sigma$ and doublet ($\phi_\pi, \phi_\eta$) will be understood as members of a nonet scalar $\phi$ [1+8 of SU(3)], and the VEV relation (1.5) will be derived by requiring that $W(\phi)$ is invariant under a Z_2 symmetry. In Sec.3, in order to give the charged lepton masses and tribimaximal neutrino mixing, we will discuss the effective Hamiltonian by assuming an Froggatt-Nelsen [13] type model. Finally, Sec.4 will be devoted to the summary and concluding remarks.

2 VEVs of SU(3) nonet scalars

The goal in the present section is to obtain the VEV relation (1.6) [i.e. (1.5)]. As seen in the previous section, in order to obtain the desirable results (1.5), we need assume an equal weight between the doublet and singlet terms of S_4. In the present paper, we assume that the S_4 symmetry is embedded into an SU(3) symmetry. The doublet ($\phi_\pi, \phi_\eta$) and singlet $\phi_\sigma$ of S_4 are embedded in the 6 and (8 + 1) of SU(3) [12]. In the present model [5], we assume that the doublet ($\phi_\pi, \phi_\eta$) and singlet $\phi_\sigma$ originate in SU(3) octet and singlet, respectively. The essential assumption in the present paper is that the fields $\phi_u$ and $\phi_d$ always appear in the theory with the form of the nonet of U(3):

$$
\phi = \begin{pmatrix}
\phi^1_1 & * & *

* & \phi^2_2 & *

* & * & \phi^3_3
\end{pmatrix},
$$

(2.1)

where

$$
\phi^1_1 = \frac{1}{\sqrt{2}} \phi_\sigma + \frac{3}{\sqrt{6}} \phi_\eta,
\phi^2_2 = \frac{1}{\sqrt{2}} \phi_\sigma - \frac{1}{\sqrt{6}} \phi_\eta + \frac{1}{\sqrt{2}} \phi_\pi,
\phi^3_3 = \frac{1}{\sqrt{2}} \phi_\sigma - \frac{1}{\sqrt{6}} \phi_\eta - \frac{1}{\sqrt{2}} \phi_\pi,
$$

(2.2)

and the index f ($f = u, d$) has been dropped.

Although we have obtained the VEV (1.5) relation from the scalar potential (1.4) by calculating $\partial V/\partial \phi_a \ (a = \pi, \eta, \sigma)$, in the present paper, we will obtain the relation (1.5) from an SU(3) invariant superpotential $W$ (The derivation of the VEV relation (1.5) from a superpotential $W$ has first been attempted by Ma [14]): The SU(3) invariant superpotential for the nonet fields $\phi_f \ (f = u, d)$ are given by

$$
W(\phi_f) = \frac{1}{2} m_f \text{Tr}(\phi_f \phi_f) + \frac{1}{2\sqrt{3}} \lambda_f \text{Tr}(\phi_f \phi_f \phi_f).
$$

(2.3)
Since, in the next section, we want to assign charges +1 and −1 of a $Z_3$ symmetry to the fields $\phi_u$ and $\phi_d$, respectively, we also assign the $Z_3$ charges +1 and −1 to the mass parameters $m_u$ and $m_d$ in Eq.(2.3), respectively. However, the U(3) invariant superpotential (2.3) cannot give the relation (1.5). As we show below, only when we drop the Tr[$(\phi^{(8)})^3$]-term in the cubic terms Tr($\phi^3$), we can obtain the VEV relation (1.5). Therefore, we introduce a $Z_2$ symmetry, and we assign the $Z_2$ parities −1 and +1 (the $Z_2$ charge +1 and 0) for the octet part $\phi^{(8)}$ and singlet part $\phi^{(1)}$ of the nonet field $\phi$, respectively. The symmetry $Z_2$ breaks U(3) into SU(3). (In other words, in the present model, the flavor symmetry U(3) is explicitly broken from the beginning by the $Z_2$ symmetry.) Under the requirement of the $Z_2$ invariance, i.e. the invariance under the transformation

$$(\phi^{(8)}, \phi^{(1)}) \rightarrow (-\phi^{(8)}, +\phi^{(1)}),$$

the terms Tr($\phi^{(8)}\phi^{(8)}\phi^{(8)}$) are forbidden. Thus, the superpotential (2.3) with the $Z_2$ invariance leads to

$$W(\phi) = \frac{1}{2} m \left[ \text{Tr}(\phi^{(8)}\phi^{(8)}) + \phi_\sigma^2 \right] + \frac{1}{2} \lambda \phi_\sigma \left[ \text{Tr}(\phi^{(8)}\phi^{(8)}) + \frac{1}{3} \phi_\sigma^3 \right]$$

$$= \frac{1}{2} m \left( \phi_\sigma^2 + \phi_\pi^2 + \phi_\eta^2 \right) + \frac{1}{2} \lambda \left[ (\phi_\pi^2 + \phi_\eta^2)\phi_\sigma + \frac{1}{3} \phi_\sigma^3 \right] + \cdots. \quad (2.5)$$

From the superpotential (2.5) with the $Z_2$ invariance, we obtain the VEV relation (1.5) as follows: From the condition

$$\frac{\partial W}{\partial (\phi^{(8)})_i^j} = m(\phi^{(8)})_i^j + \lambda \phi_\sigma (\phi^{(8)})_i^j = 0,$$

we obtain

$$m + \lambda \phi_\sigma = 0,$$

for $(\phi^{(8)})_i^j \neq 0$. By eliminating $m$ from Eq.(2.7) and the condition

$$\frac{\partial W}{\partial \phi_\sigma} = m\phi_\sigma + \frac{1}{2} \lambda \left[ \text{Tr}(\phi^{(8)}\phi^{(8)}) + \phi_\sigma^2 \right] = 0,$$

we obtain the relation

$$\phi_\sigma^2 = \text{Tr}(\phi^{(8)}\phi^{(8)}) = \phi_\pi^2 + \phi_\eta^2 + \cdots,$$

where “…” denotes the contributions of 3 and 3’ of $S_4$.

The result (2.9) is still not our goal, because the relation contains the VEVs of the 3 and 3’ of $S_4$. So far, we have not discussed the splitting among the $S_4$ multiplets. Now, we bring a soft symmetry breaking of SU(3) into $S_4$ with an infinitesimal parameter $\varepsilon$ into the mass term of $W(\phi)$ as

$$\text{Tr}(\phi^{(8)}\phi^{(8)}) \Rightarrow \phi_\pi\phi_\pi + \phi_\eta\phi_\eta + (1 + \varepsilon) \sum_{i \neq j} (\phi^{(8)})_i^j(\phi^{(8)})_j^i,$$

(2.10)
by hand. Recall that when we obtain the relation (2.7), we have assumed $(\phi^{(8)})^i_j \neq 0$. Now, the conditions (2.6) are modified into the following conditions:

\[(1 + \varepsilon)m + \lambda \phi_a \] $(\phi^{(8)})^i_j = 0 \quad (i \neq j), \quad (2.11)$
\[m + \lambda \phi_a \] $\phi_a = 0 \quad (a = \pi, \eta), \quad (2.12)$

Therefore, we must take either $(\phi^{(8)})^i_j = 0 \quad (i \neq j)$ or $\phi_a = 0 \quad (a = \pi, \eta)$ for $\varepsilon \neq 0$. When we choose the solution

\[((\phi^{(8)})^i_j) = 0 \quad (i \neq j), \quad (2.13)\]

we can obtain the desirable relation (1.5). (However, it is possible that we can also take another solution with $\phi_\pi = \phi_\eta = 0$ and $(\phi^{(8)})^i_j \neq 0$. The VEV solutions are not unique. The result (1.5) is merely one of the possible solutions.)

Thus, we have obtained not only the desirable VEV relation (1.5), but also the results (2.13). It should be worthwhile noticing that if we have assume the superpotential (2.3) without requiring the $Z_2$ invariance, we could obtain neither (1.5) nor (2.13).

By the way, we know that the three masses in any sectors of quarks and leptons are completely different among them. Therefore, if we assume a flavor symmetry, the symmetry must finally be broken completely. Usually, a relation which we derive in the exact symmetry limit is only approximately satisfied under the symmetry breaking. Although we derive the VEV relation (1.5) under the $S_4$ symmetry, the problem is whether the VEV relation (1.5) which is obtained under the $S_4$ symmetry is spoiled or not when we introduce such a symmetry breaking. In Ref.[5], we will find that such a symmetry breaking term without spoiling the relation (1.5) is indeed possible.

3 Effective Hamiltonian

If we regard the scalars $\phi_u$ and $\phi_d$ as SU(2)$_L$ doublets, such a model with multi-Higgs doublets causes a flavor changing neutral current (FCNC) problem. Therefore, we must consider that the fields $\phi_u$ and $\phi_d$ are SU(2)$_L$ singlets. In the present paper, we assume a Froggatt-Nielsen [13] type model

\[H^{\text{eff}} = y_\ell \ell_L H^d_L \phi_d \xi e_R + y_\nu \ell_L H^u_L \phi_u \chi \nu_R + y_R \nu_R \Phi_R \nu_R^*, \quad (3.1)\]

where $\ell_{iL}$ are SU(2)$_L$ doublet leptons, $\ell_{iL} = (\nu_{iL}, e_{iL})$, $H^d_L$ and $H^u_L$ are conventional SU(2)$_L$ doublet Higgs scalars, $\phi_f \ (f = u, d)$, $\xi$ and $\chi$ are SU(2)$_L$ singlet scalars, and $\Lambda$ is a scale of the effective theory. We consider that $\langle \phi_f \rangle / \Lambda$, $\langle \xi \rangle / \Lambda$ and $\langle \chi \rangle / \Lambda$ are of the order of 1. The scalar $\Phi_R$ has been introduced in order to generate the Majorana mass $M_R$ of the right-handed neutrino $\nu_R$. As we note later, in the present model, the right-handed neutrinos $\nu_R = (\nu_R^+ + \nu_R^-) / \sqrt{2}$ are singlets of the SU(3) flavor. The role of $\xi = (\xi^+ + \xi^-) / \sqrt{2}$ and $\chi$ will be explained later. In order to understand the appearance of the combinations $H^d_L \phi_d \phi_d \xi$ and $H^u_L \phi_u \chi$, we assume two $Z_3$ symmetries ($Z_3$ and $Z'_3$ in Table 1). Those quantum number assignments are given in Table 1. However, even with those quantum numbers, we cannot distinguish the state $\phi_f^\dagger$ from $\phi_f \phi_f$. For example, the interaction $\ell_L H^{\dagger}_\theta \phi_d \xi e_R$ is possible in addition to the interaction
Table 1  SU(3) and $S_4$ assignments of the fields

<table>
<thead>
<tr>
<th>Fields</th>
<th>SU(2)$_L$</th>
<th>SU(3)</th>
<th>$S_4$</th>
<th>$Z_3$</th>
<th>$Z'_3$</th>
<th>$Z_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ell_L$</td>
<td>2</td>
<td>3</td>
<td>$3'$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$e_R$</td>
<td>1</td>
<td>3</td>
<td>$3'$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\nu_R^{(\pm)}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0/±1</td>
</tr>
<tr>
<td>$\phi_u$</td>
<td>1</td>
<td>1+$8$</td>
<td>$1+(2+3+3')$</td>
<td>+1</td>
<td>+1</td>
<td>0/±1</td>
</tr>
<tr>
<td>$\phi_d$</td>
<td>1</td>
<td>1+$8$</td>
<td>$1+(2+3+3')$</td>
<td>−1</td>
<td>−1</td>
<td>0/±1</td>
</tr>
<tr>
<td>$\xi^{(\pm)}$</td>
<td>1</td>
<td>3</td>
<td>$3'$</td>
<td>+1</td>
<td>−1</td>
<td>0</td>
</tr>
<tr>
<td>$H^u_L$</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>+1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$H^d_L$</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>−1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\Phi_R$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Although we have started from an SUSY scenario in the previous section, now, we have adopted an effective Hamiltonian which is not renormalizable. Therefore, in principle, the interaction $\bar{\ell}_L H_d \phi_d \xi e_R$. Although we have started from an SUSY scenario in the previous section, now, we have adopted an effective Hamiltonian which is not renormalizable. Therefore, in principle, the interaction $\bar{\ell}_L H_d \phi_d \xi e_R$ cannot be ruled out. For the moment, in order to forbid such an undesirable term, we assume that the fields which can appear in the effective Hamiltonian are confined to holomorphic ones.

(a) Charged lepton sector

Recall that we have already assumed the invariance of the superpotential under the $Z_2$ transformation (2.4) in order to drop the cubic part of the octet $\phi^{(8)}$. Therefore, the term $\phi \phi$ means $\phi^{(8)} \phi^{(8)} + \phi^{(1)} \phi^{(1)}$ under the $Z_2$ invariance. However, in order to give $m_{ei} \propto \langle \phi_i \rangle^2$, what we want is not $\phi^{(8)} \phi^{(8)} + \phi^{(1)} \phi^{(1)}$, but $\phi^{(8)} \phi^{(8)} + \phi^{(1)} \phi^{(1)} + \phi^{(8)} \phi^{(1)} + \phi^{(1)} \phi^{(8)}$. In order to evade this problem, we introduce additional fields $\xi^{(+)}$ and $\xi^{(-)}$ whose $Z_2$ parity are $+1$ and $-1$, respectively. The effective interactions in the charged lepton sector are given by

$$H_{e}^{\text{eff}} = \frac{y_{e}}{\sqrt{2}} \bar{\ell}_L (\phi^{(8)} d^{(8)} + \phi^{(1)} d^{(1)}) (\xi^{(+)} + \xi^{(-)}) e_R,$$

where we have dropped the Higgs scalar $H^d_L$ since we discuss flavor structure only. The expression (3.2) becomes

$$H_{e}^{\text{eff}} = \frac{y_{e}}{\sqrt{2}} \bar{\ell}_L [(\phi^{(8)} d^{(8)} + \phi^{(1)} d^{(1)}) \xi^{(+) + \xi^{(-)}} + (\phi^{(8)} d^{(8)} + \phi^{(1)} d^{(1)}) \xi^{(-)}] e_R. \quad (3.3)$$

Since we have assumed that $\xi^{(+) + \xi^{(-)}}$, we also assume

$$\langle \xi^{(+) + \xi^{(-)}} \rangle \equiv v_{\xi}. \quad (3.4)$$

Then, we obtain the effective Hamiltonian for the charged leptons

$$H_{e}^{\text{eff}} = \frac{y_{e} v_{d} v_{\xi}}{\sqrt{2} \Lambda^3} \sum_{i} \bar{\ell}_L (\phi^{(8+1)} d^{(1)}) e_{Ri}, \quad (3.5)$$
where \(v_d = \langle H_L^d \rangle\). Since the fields \((\phi_d)_d^i\) are defined by Eq. (2.2), we can obtain the charged lepton mass relation (1.1) from the VEV relation (1.6).

(b) Neutrino sector

In the present model, the right-handed neutrinos \(\nu^{(\pm)}\) are singlets of SU(3). Therefore, in the neutrino seesaw mass matrix \(M_\nu = m_\nu^2 M_R^{-1} (m_L^\nu)^T\), \(M_R\) is a 1 \times 1 matrix and \(m_L^\nu\) is a 3 \times 1 matrix. In order to compensate for the absence of the conventional triplet neutrinos \(\nu_R\), a new scalar \(\chi\) which is a triplet of SU(3) has been introduced. The neutrino Dirac mass terms are given by the following effective Hamiltonian

\[
H_{\text{Dirac}}^{\text{eff}} = y_\nu \frac{v_u}{\Lambda^2} \bar{\nu}_L^i \langle (\phi_u)_i^j \rangle \langle \chi \rangle \left( \nu_R^{(+)} + \nu_R^{(-)} \right),
\]

where \(v_u = \langle H_L^u \rangle\). It is likely that the scalar potential \(V(\chi)\) for the SU(3) triplet \(\chi\) has a specific VEV solution

\[
\langle \chi_1 \rangle = \langle \chi_2 \rangle = \langle \chi_3 \rangle = v_\chi.
\]

When we assume the VEVs (3.7), we obtain

\[
H_{\text{Dirac}}^{\text{eff}} = y_\nu \frac{v_u v_\chi}{\sqrt{2} \Lambda^2} \bar{\nu}_\eta \nu_\sigma \bar{\nu}_\pi L \left[ \begin{array}{c} v_\eta \\ 0 \\ v_\pi \end{array} \right] \left[ \begin{array}{c} \nu_R^{(-)} \\ \nu_R^{(+)} \\ 0 \end{array} \right],
\]

where \(v_u = \langle \phi_u^a \rangle\) \((a = \eta, \pi, \sigma)\) (for convenience, we have dropped the index \(u\)). Therefore, we obtain the effective neutrino mass matrix on the \((\eta, \sigma, \pi)\) basis,

\[
U_T^T M_\nu U_T = M_\nu^{(\eta \sigma \pi)} = \frac{1}{M_R^{(-)}} \left( \begin{array}{ccc} v_\eta^2 & 0 & v_\pi v_\eta \\ 0 & v_\pi v_\eta & 0 \\ v_\pi v_\eta & 0 & v_\pi^2 \end{array} \right) + \frac{1}{M_R^{(+)}} \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & v_\sigma^2 & 0 \\ 0 & 0 & 0 \end{array} \right),
\]

where \(M_R^{(\pm)} = y_R^{(\pm)} \langle \Phi_R \rangle\), and we have dropped the common factors \((y_\nu v_u v_\chi / \sqrt{2} \Lambda^2)^2\). By the way, the ratio \(v_\pi / v_\eta\) cannot be determined from the potential (2.6), and the ratio is determined by a soft \(S_4\) symmetry breaking term \(W_{SB}\) which has been discussed in the previous section. We can choose a solution \(v_\pi = 0\) in the superpotential \(W(\phi_u)\) by adjusting the parameter \(\beta\) in \(W_{SB}\), differently from the case of \(W(\phi_d)\).

Then, the neutrino mass matrix (3.9) becomes a diagonal form \(D_\nu = (1/M_R^{(-)}) \text{diag}(v_\eta^2, 0, 0) + (1/M_R^{(+)}) \text{diag}(0, v_\sigma^2, 0)\). Since the mass matrix \(M_\nu\) on the \((\nu_1, \nu_2, \nu_3) = (\nu_e, \nu_\mu, \nu_\tau)\) basis is given by

\[
M_\nu = U_T^T \nu^{(\eta \sigma \pi)} U_T = U_T D_\nu U_T^T,
\]

we can obtain the tribimaximal mixing

\[
U_\nu = U_T B,
\]

and the neutrino masses

\[
m_{\nu1} = k v_\eta^2, \quad m_{\nu2} = k v_\sigma^2, \quad m_{\nu3} = 0,
\]
for the case of $M_R^{(+)} = M_R^{(-)} \equiv M_R$, where $k = (y_\nu v_\alpha v_\chi)^2/2M_R \Lambda^4$ and $(\nu_\eta, \nu_\sigma, \nu_\pi)$ has been renamed $(\nu_1, \nu_2, \nu_3)$ according to the conventional naming.

However, since we have taken $v_\pi = 0$, the value of $v_\eta$ satisfies $v_\eta^2 = v_\sigma^2$ from the relation (1.5), so that the result (3.12) gives $m_{\nu 1} = m_{\nu 2}$. The observed value [15] $\Delta m^2_{\text{solar}}$ is small, but it is not zero. Therefore, we must consider a small deviation between the first and second terms in (3.9) (i.e. $M_R^{(+)} \neq M_R^{(-)}$). Since the value $M_R^{(-)}$ is free in the present model, we cannot predict an explicit value of the ratio $\Delta m^2_{\text{solar}}/\Delta m^2_{\text{atm}}$.

Since the present model gives an inverse hierarchy of the neutrino masses, the predicted effective electron neutrino mass

$$\langle m_{\nu_e} \rangle = \sum_i U_{ei}^2 m_{\nu_i} \simeq |m_{\nu 1}| \simeq |m_{\nu 2}| \simeq \sqrt{\Delta m^2_{\text{atm}}} = 5.23^{+0.25}_{-0.40} \times 10^{-2} \text{eV},$$

(3.13)

where we have used the value [16] $\Delta m^2_{\text{atm}} = 2.74^{+0.44}_{-0.26} \times 10^{-3} \text{eV}^2$. This value (3.13) is sufficiently sensitive to the next generation experiments of the neutrinoless double beta decay.

4 Summary

In conclusion, on the basis of the $S_4$ symmetry which is embedded into SU(3), we have investigated a lepton mass model with the effective Hamiltonian of the Froggatt-Nielsen type (3.1). We have assumed that the singlet and doublet of $S_4$ originate in the singlet and octet of SU(3), and we have obtained the VEV relation (1.5). In the derivation of the VEV relation (1.5), the essential assumptions for the superpotential $W(\phi_f)$ are the following two: (i) the scalar fields $\phi_f$ always appear in terms of the nonet form (2.1) of U(3); (ii) the superpotential $W(\phi_f)$ is invariant under the $Z_2$ transformation (2.4). Then, we have obtained not only the VEV relation (1.5), but also $\langle (\phi^{(8)}_i)^2 \rangle = 0$ (i $\neq$ j) for the other components of $\phi^{(8)}$ (i.e. $\langle 3 \rangle = \langle 3' \rangle = 0$).

In the charged lepton sector, we have assumed the Froggatt-Nielsen-type effective Hamiltonian $\bar{e}_L^i (\phi_d)_i^j (\phi_d)_j^j e_{Rj}$. Since we have obtained the VEV of $\phi_d^i$

$$\langle \phi_d \rangle = \text{diag} \left( \frac{1}{\sqrt{3}} v_\sigma + \frac{2}{\sqrt{6}} v_\eta, \frac{1}{\sqrt{3}} v_\sigma - \frac{1}{\sqrt{6}} v_\eta, \frac{1}{\sqrt{2}} v_\pi, \frac{1}{\sqrt{3}} v_\sigma - \frac{1}{\sqrt{6}} v_\eta, \frac{1}{\sqrt{2}} v_\pi \right),$$

(4.1)

we can obtain

$$m_{ei} \propto \langle (\phi_d)_i^j \rangle^2 \equiv (v_i^j)^2,$$

(4.2)

so that the charged lepton masses $m_{ei}$ satisfy the relation (1.1) under the definition (2.2). Here, we would like to emphasize that the result $\langle \phi_d^j \rangle = 0$ for $i \neq j$, Eq.(2.13), is essential to obtain Eq.(4.2) in the Froggatt-Nielsen-type model. In the Froggatt-Nielsen-type model, if we had considered a triplet scalar $\phi$, we could obtain the result $m_{ei} \propto v_i^2$, because the effective interaction $\bar{e}_L^i \phi e_{Rj}$ gives $\bar{e}_L^i (\phi_i^j e_{Rj})$. On the other hand, if we had adopted a seesaw-type model, by assuming a triplet scalar $\phi$ whose effective interaction is given by

$$H_e = \sum_i \left( \bar{e}_{Li}^i \phi_i E_{Ri}^i + \bar{E}_{Li}^i \phi_i e_{Ri}^i + M_E \bar{E}_{Li}^i E_{Ri}^i \right),$$

(4.3)
we could automatically obtain a form

\[ H^\text{eff} = \sum_i \frac{1}{M_E} \bar{e}_{Li} \phi_i^2 e_{Ri}, \] (4.4)

through a seesaw mechanism [17]. However, it is not easy to obtain the VEV relation (1.6) for the triplet scalar \( \phi \). This is the main motive for introducing the nonet (not triplet) scalar \( \phi \).

For the neutrino sector, we have obtained the tribimaximal mixing (1.2) by introducing an SU(3) triplet scalar \( \chi \) and the two SU(3) singlet right-handed neutrinos \( \nu_R^{(\pm)} \) in addition to the nonet scalar \( \phi_u \). In the present model, the right-handed neutrinos \( \nu_R^{(\pm)} \) are singlets of SU(3), the Majorana neutrino mass matrices \( M_R^{(\pm)} \) have no flavor structure, (i.e. \( M_R \) are 1 matrices). Also note that the neutrino Dirac mass matrix \( m_{\nu_L} \) is a 3 \( \times \) 1 matrix. This plays an essential role to derivation of the tribimaximal mixing. For the neutrino mass spectrum, since the model gives \( m_{\nu_1} = m_{\nu_2} \) in the limit of \( M_R^{(\pm)} = M_R^{(-)} \), we must consider a small deviation \( M_R^{(\pm)} \neq M_R^{(-)} \). Since the value of \( M_R^{(-)} / M_R^{(\pm)} \) is a free parameter in the present model, we cannot predict the value \( \Delta m^2_{\text{solar}} / \Delta m^2_{\text{atm}} \) at present, although the smallness of the ratio \( \Delta m^2_{\text{solar}} / \Delta m^2_{\text{atm}} \) can be understood. Since the present model gives an inverse hierarchy of the neutrino masses, we can predict the effective electron neutrino mass \( \langle m_{\nu_e} \rangle \simeq 0.05 \) eV, which is sufficiently sensitive to the next generation experiments of the neutrinoless double beta decay.

The present model seems to provide suggestive hints on seeking for a model which leads to the tribimaximal mixing (1.2) and the charged lepton mass relation (1.1), although the model has still many points which should be improved. We summarize some of the future tasks:

(i) Seeking for a more natural mechanism which can provide \( m_{ei} \propto v^2_i \), apart from a Froggatt-Nielsen-type model.

(ii) Seeking for a model which is completely predictable neutrino mass spectrum (in the present model, \( \Delta m^2_{21} \) was not predictable) together with the nearly tribimaximal mixing.

(iii) In the present model, the right-handed neutrino \( \nu_R \) was a singlet of SU(3). However, it is likely that \( \nu_R \) still a triplet.

(iv) Seeking for the origin of the symmetry breaking of SU(3) into S\(_4\).

We hope that the present model will also provide a promising clue to the unified mass matrix model of the quarks and leptons.

**Acknowledgments**

The author would like to thank E. Takasugi, H. Fusaoka and N. Haba for helpful conversations. This work is supported by the Grant-in-Aid for Scientific Research, Ministry of Education, Science and Culture, Japan (No.18540284).

**References**


