# Systematic analysis of the determination of sgn $\delta m^2_{31}$ and $\delta_{CP}$ in LBL $\nu$OscExp

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# Neutrino Oscillation

- Evolution equation of neutrinos

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\nu_{\mathrm{e}}\\ \nu_{\mu}\\ \nu_{\tau}\end{pmatrix} = \frac{1}{2E}\left[U\begin{pmatrix}0 & & \\ & \delta m^2_{21} & \\ & & \delta m^2_{31}\end{pmatrix}U^{\dagger} + \begin{pmatrix}a & & \\ & 0 & \\ & & 0\end{pmatrix}\right]\begin{pmatrix}\nu_{\mathrm{e}}\\ \nu_{\mu}\\ \nu_{\tau}\end{pmatrix} \equiv H\begin{pmatrix}\nu_{\mathrm{e}}\\ \nu_{\mu}\\ \nu_{\tau}\end{pmatrix}$$

  - Quadratic mass differences

$$\delta m^2_{ij} \equiv m^2_i - m^2_j$$

$$\delta m^2_{21} \simeq 8\times 10^{-5}\,\mathrm{eV}^2\,,$$

$$\left|\delta m^2_{31}\right| \simeq (2 - 2.5)\times 10^{-3}\,\mathrm{eV}^2$$

(Fogli *et al.* 2005)

$$\begin{cases}\delta m^2_{31} > 0 & \text{“Normal” hierarchy}\\ \delta m^2_{31} < 0 & \text{“Inverted” hierarchy}\end{cases}$$

  - Mixing matrix, mixing angles and CP phase(s)

$$U = \begin{pmatrix}1 & 0 & 0\\ 0 & c_{23} & s_{23}\\ 0 & -s_{23} & c_{23}\end{pmatrix}\begin{pmatrix}c_{13} & 0 & s_{13}\mathrm{e}^{-\mathrm{i}\delta_{\mathrm{CP}}}\\ 0 & 1 & 0\\ -s_{13}\mathrm{e}^{\mathrm{i}\delta_{\mathrm{CP}}} & 0 & c_{13}\end{pmatrix}\begin{pmatrix}c_{12} & s_{12} & 0\\ -s_{12} & c_{12} & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & & \\ & \mathrm{e}^{\mathrm{i}\phi_1} & \\ & & \mathrm{e}^{\mathrm{i}\phi_2}\end{pmatrix}$$

$$c_{ij} \equiv \cos\theta_{ij} \qquad s_{ij} \equiv \sin\theta_{ij}$$

$$\sin^2\theta_{12} \sim 0.3\,,\quad \sin^2\theta_{23} \sim 0.45\,,\quad \sin^2\theta_{13} \lesssim 0.04$$

(Fogli *et al.* 2005)

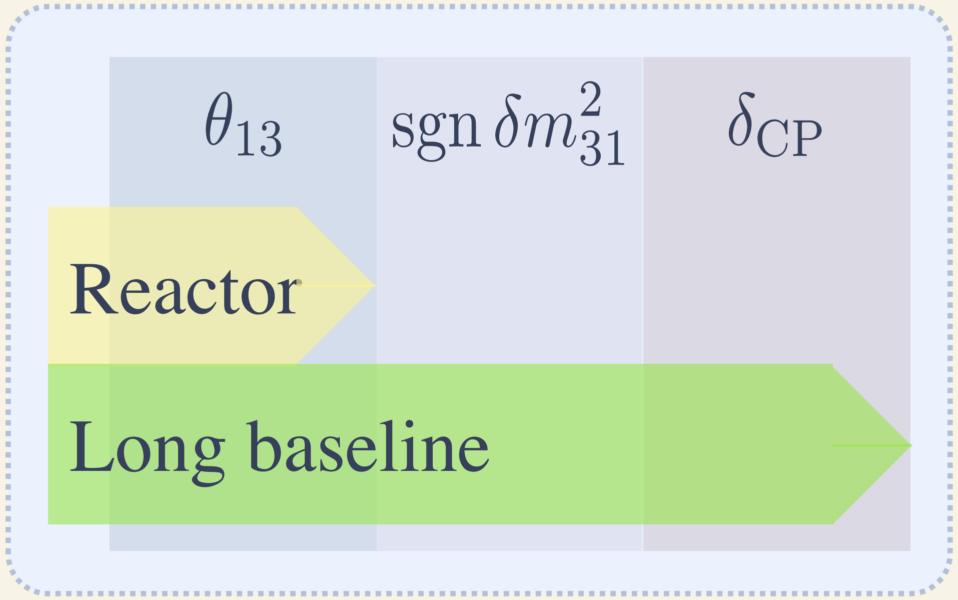
  - Matter effect

$$a = 2\sqrt{2}G_{\mathrm{F}}n_{\mathrm{e}}E = 7.63\times 10^{-5}\,\mathrm{eV}^2\,\frac{\rho}{\mathrm{g\,cm}^3}\,\frac{E}{\mathrm{GeV}}$$

  - Unknowns to date: $\theta_{13}\,,$ $\mathrm{sgn}\,\delta m^2_{31}\,,\ \delta_{\mathrm{CP}}\,,$ $\phi_1, \phi_2$

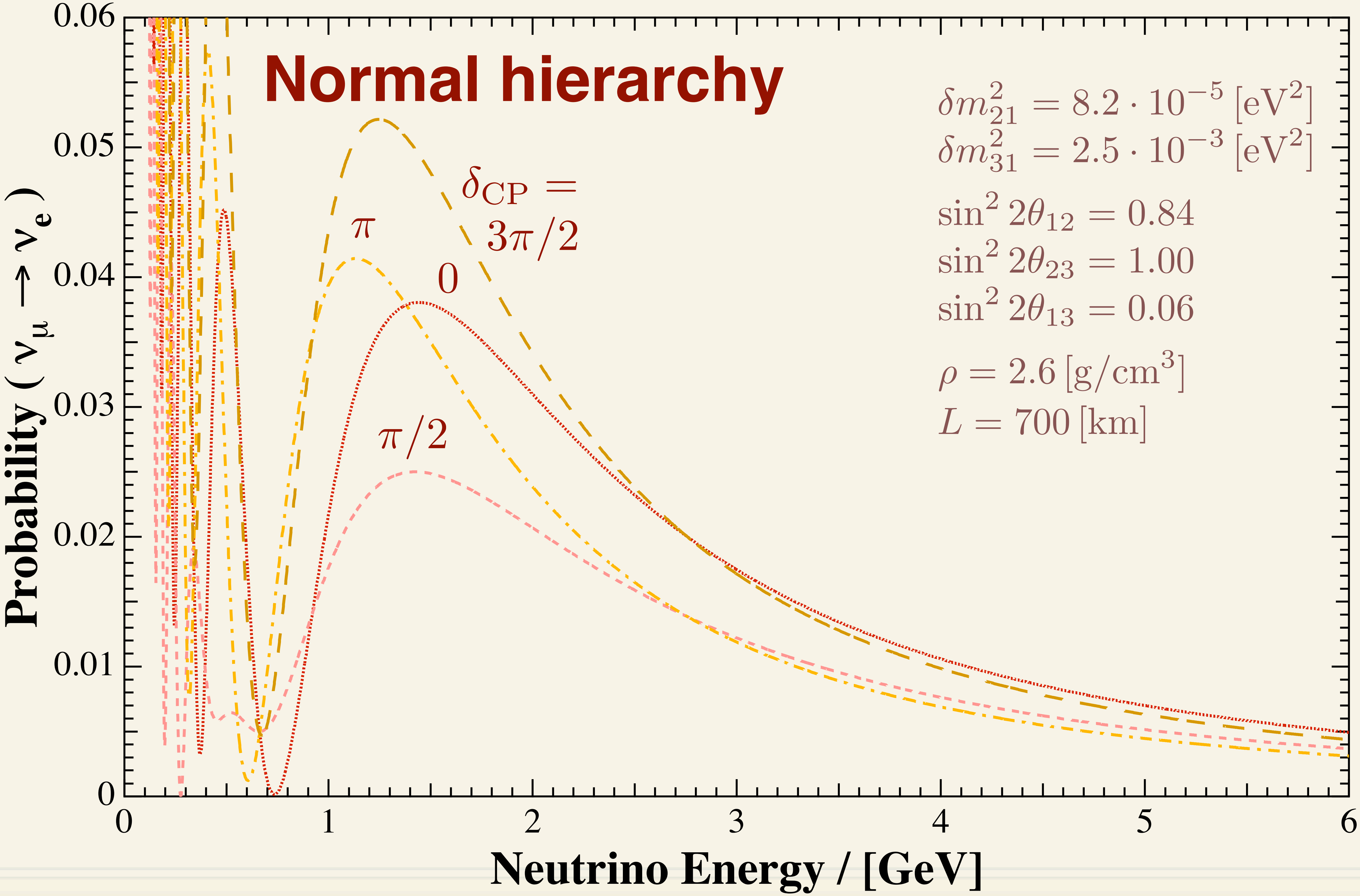
# Scope of this Talk

- Situations considered
  - $\theta_{13}$ constrained in advance of the future generation of long-baseline experiment.
  - The value of $\theta_{13}$ is not too small so that the *CP*-violation effect is accessible. $(\sin^2 2\theta_{13} \gtrsim 0.01)$

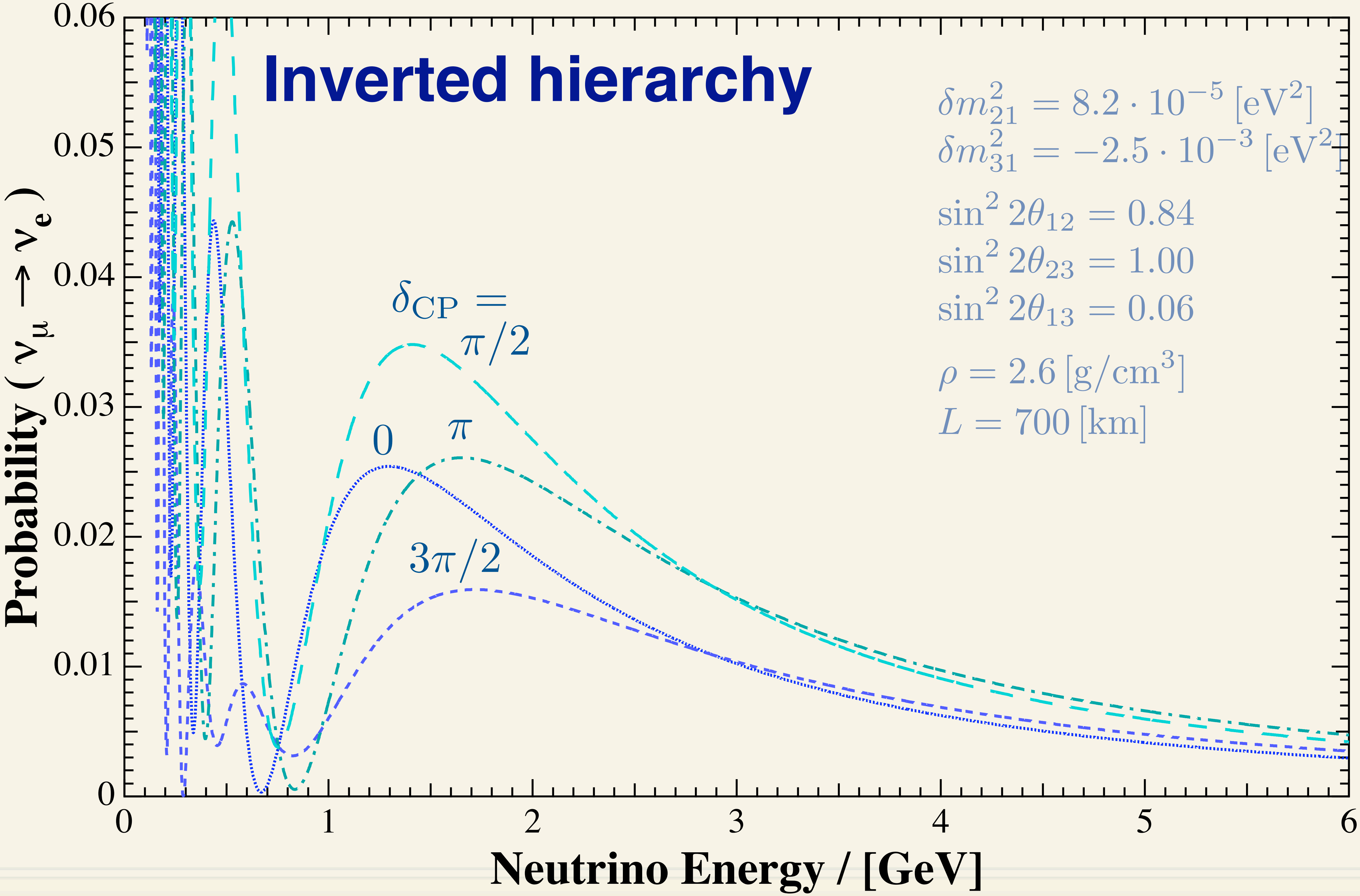



- Purpose
  - Provide a perspective of the presence and absence of the degeneracies regarding hierarchy $(\mathrm{sgn}\,\delta m^2_{31})$ and $\delta_{\rm CP}$ .
  - Discuss the way to get out of this degeneracy.

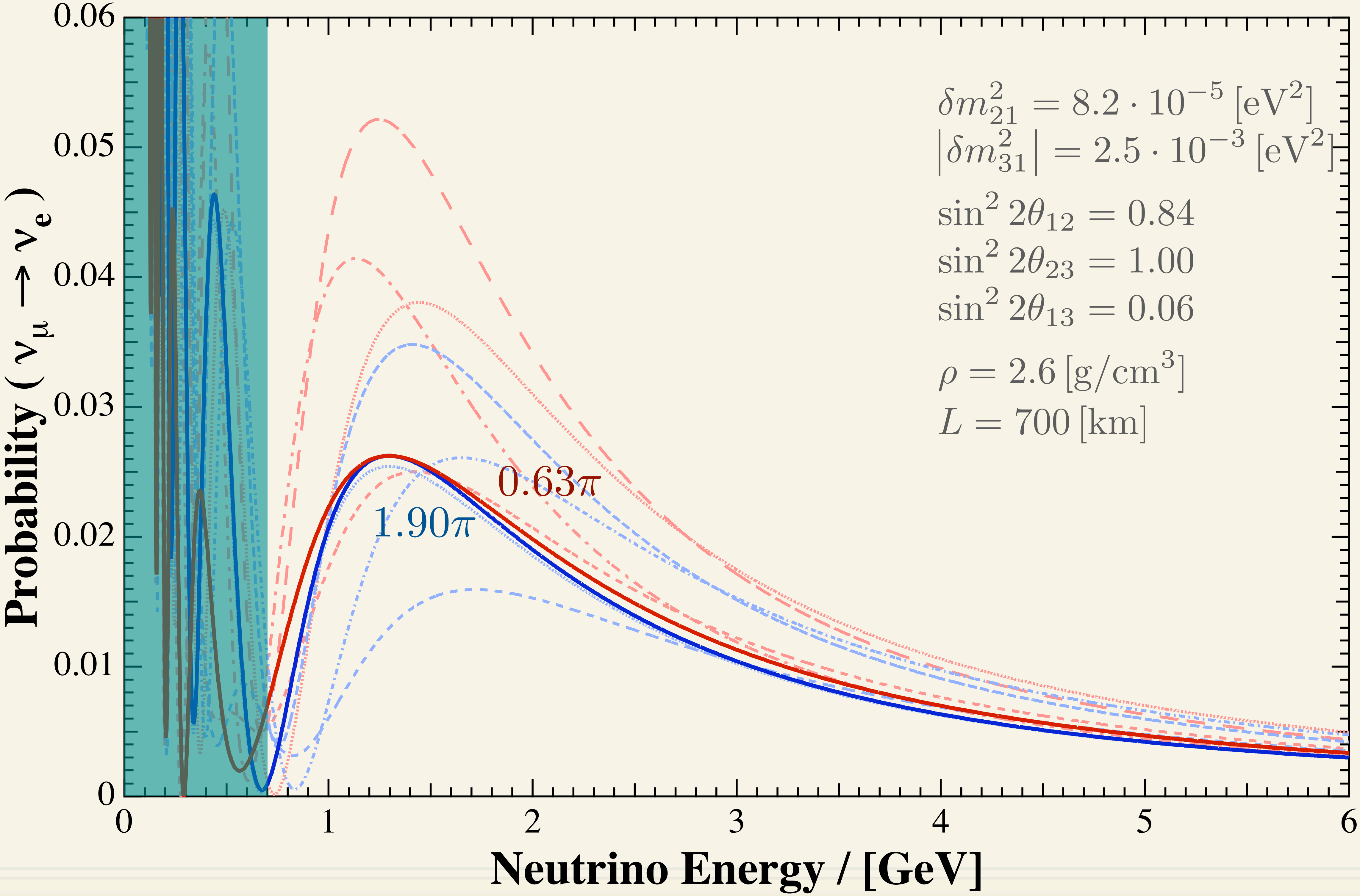
# Oscillation Probabilities



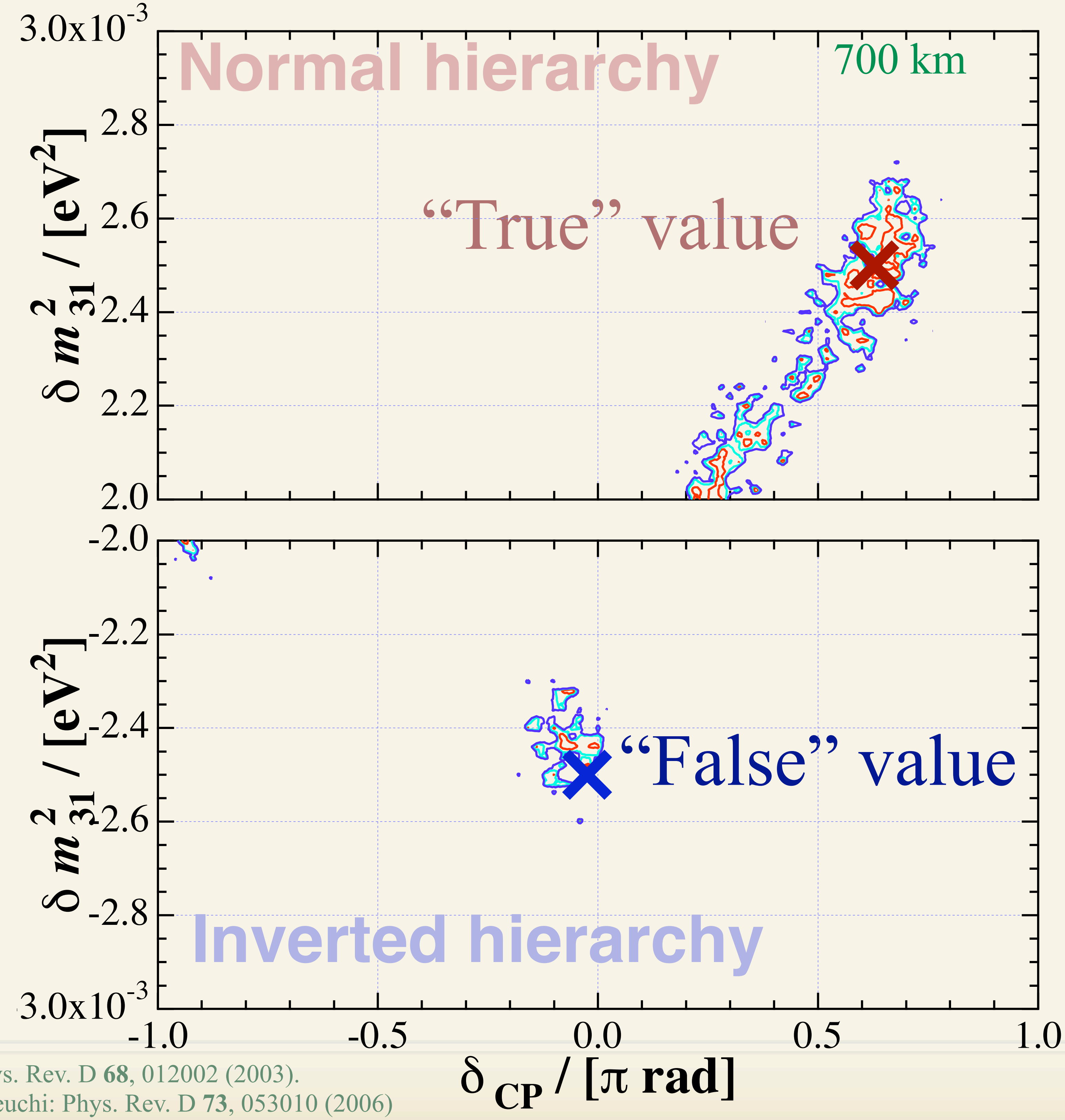
# Oscillation Probabilities



# Oscillation Probabilities

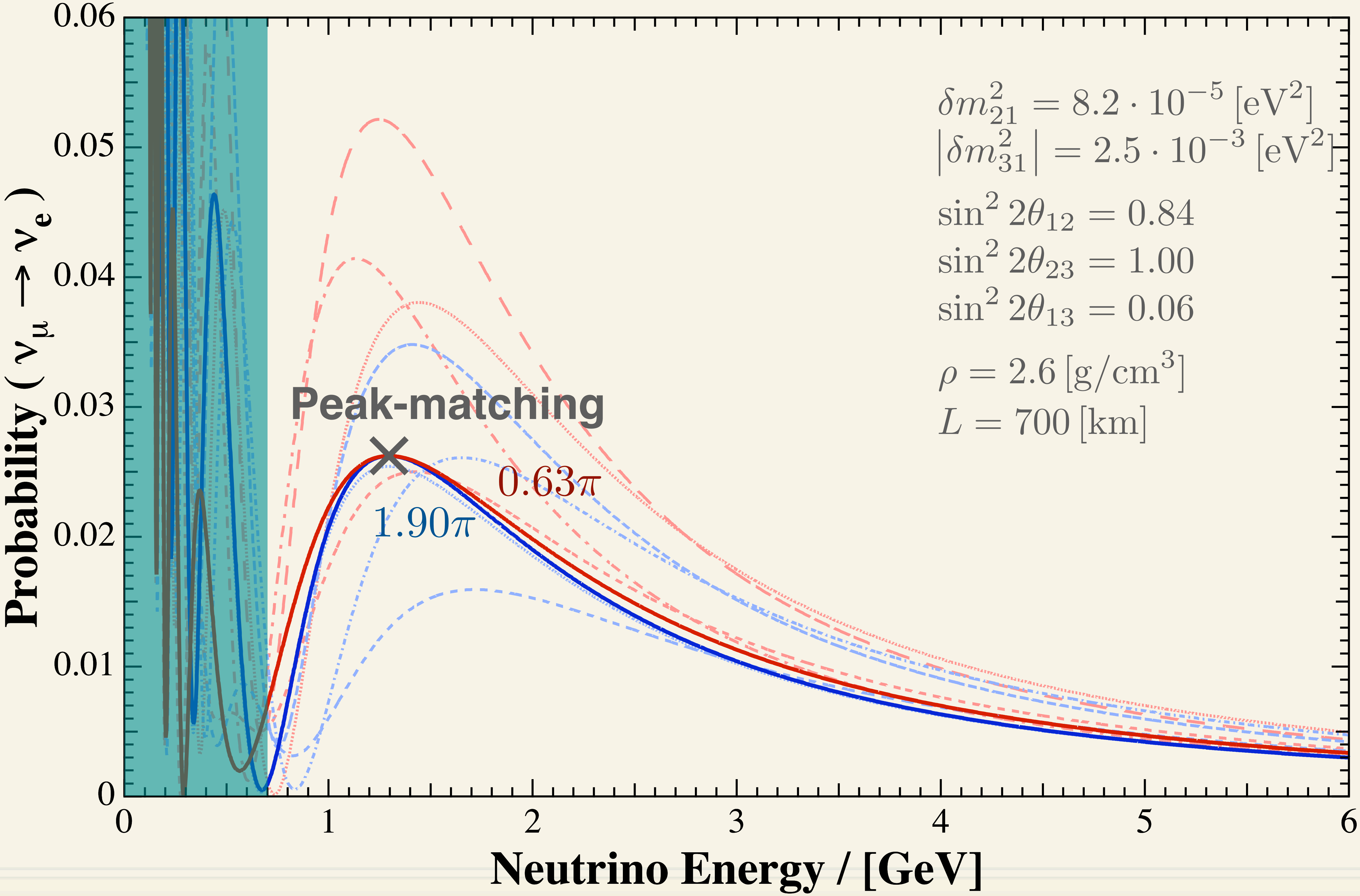


# Hierarchy Degeneracy




Beam flux: M. V. Diwan *et al*., Phys. Rev. D **68**, 012002 (2003).
Analysis: MK, Okamura, MS, Takeuchi: Phys. Rev. D **73**, 053010 (2006)

# Oscillation Peaks



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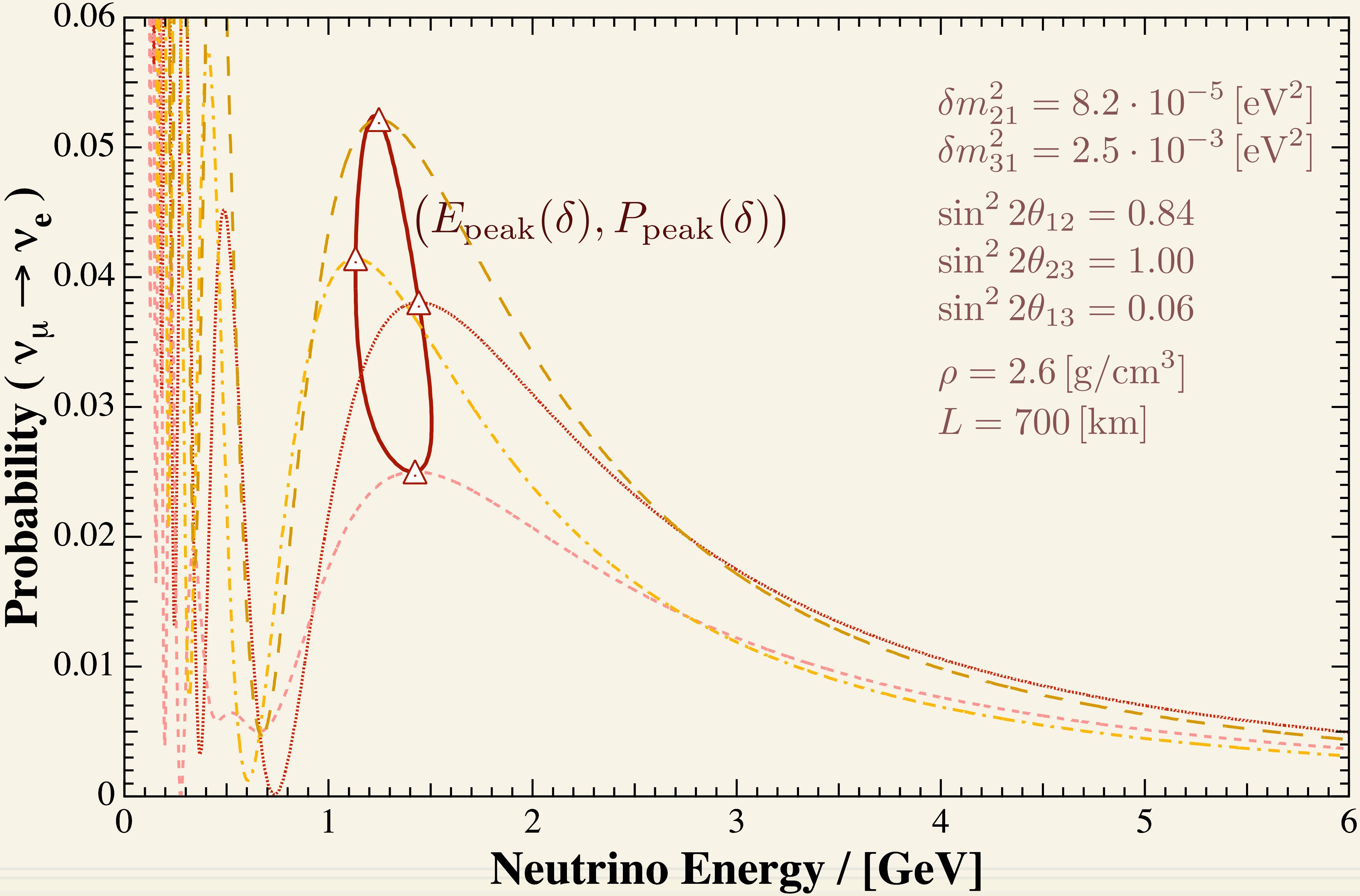

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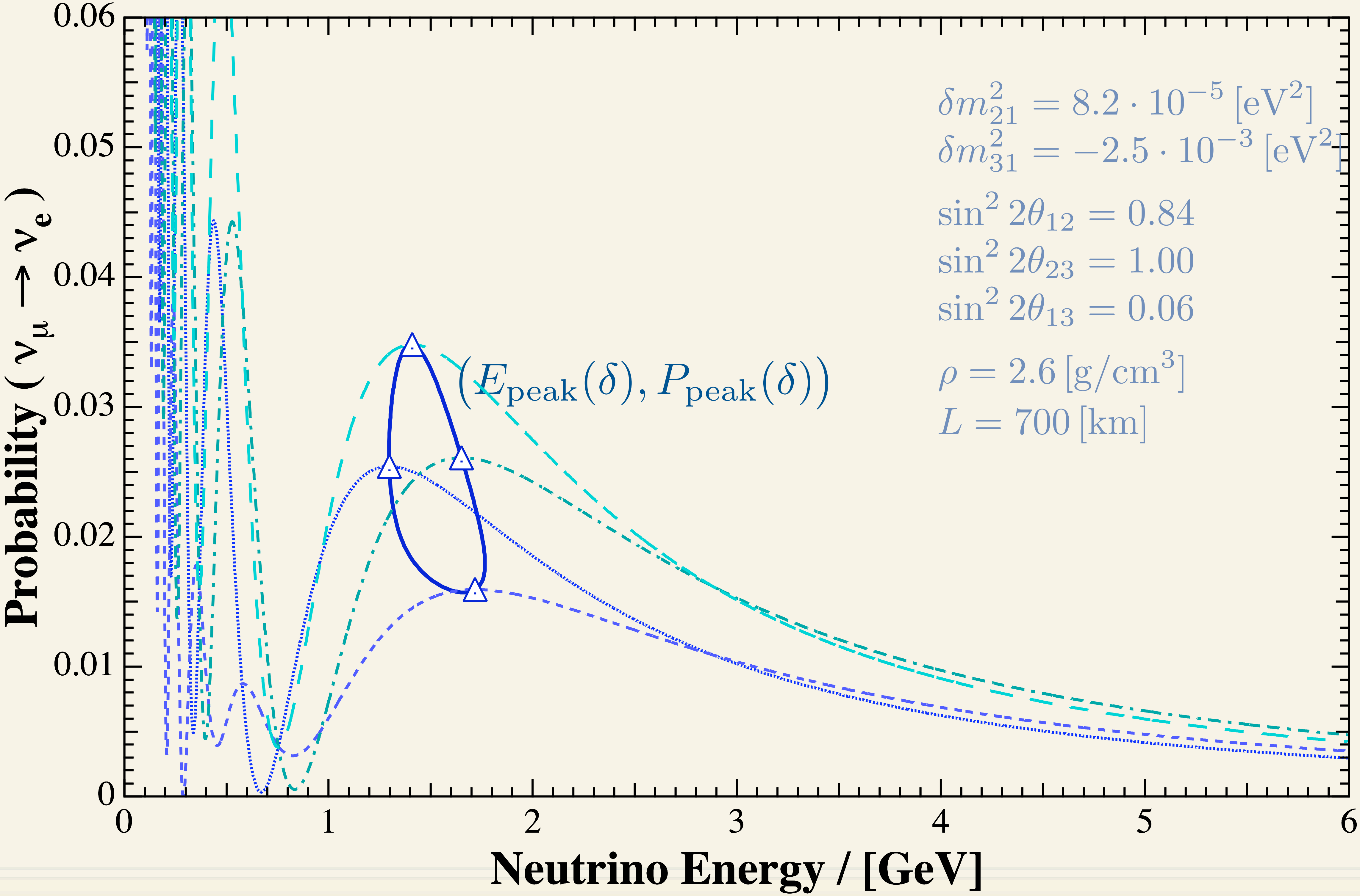

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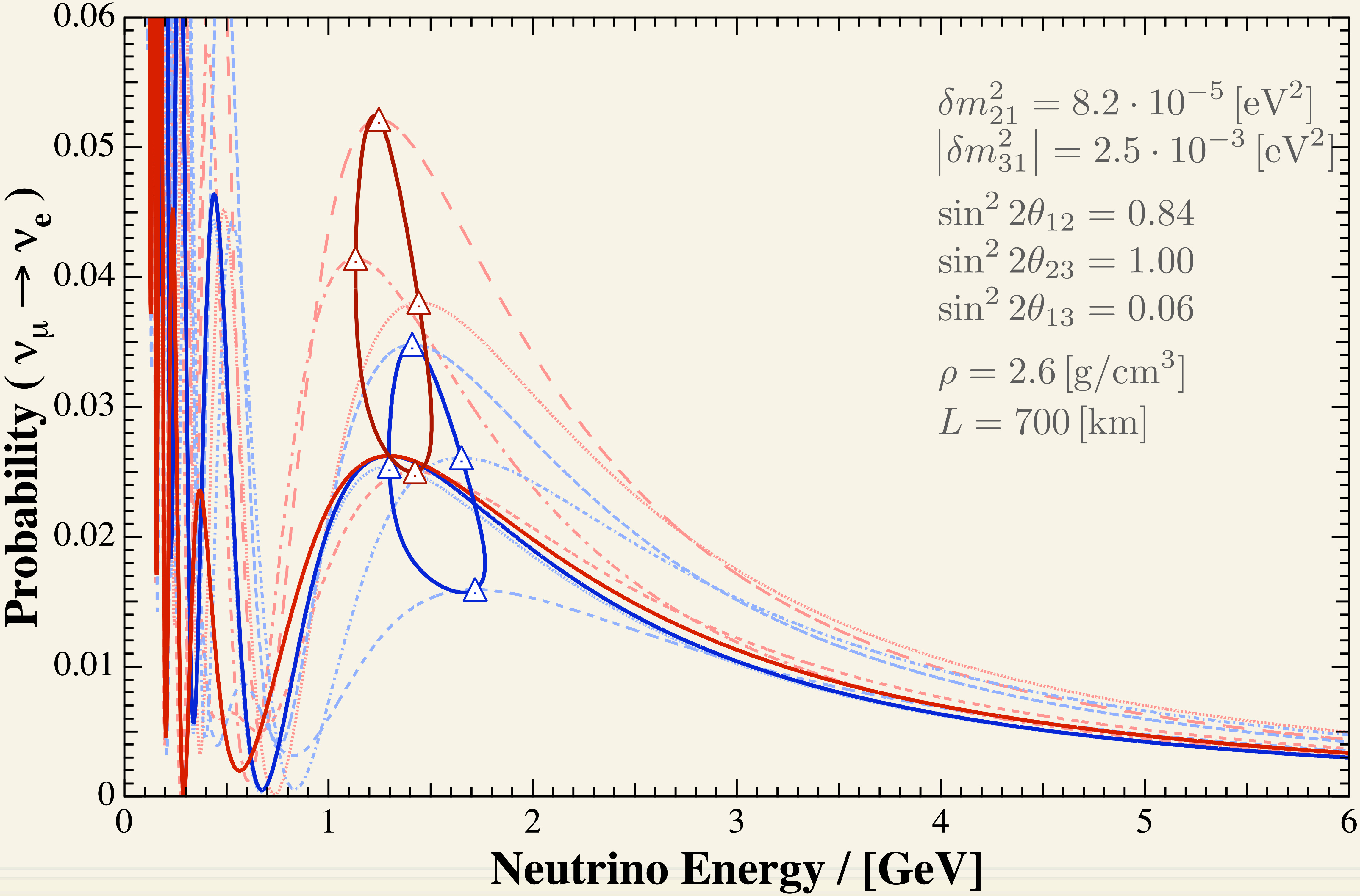
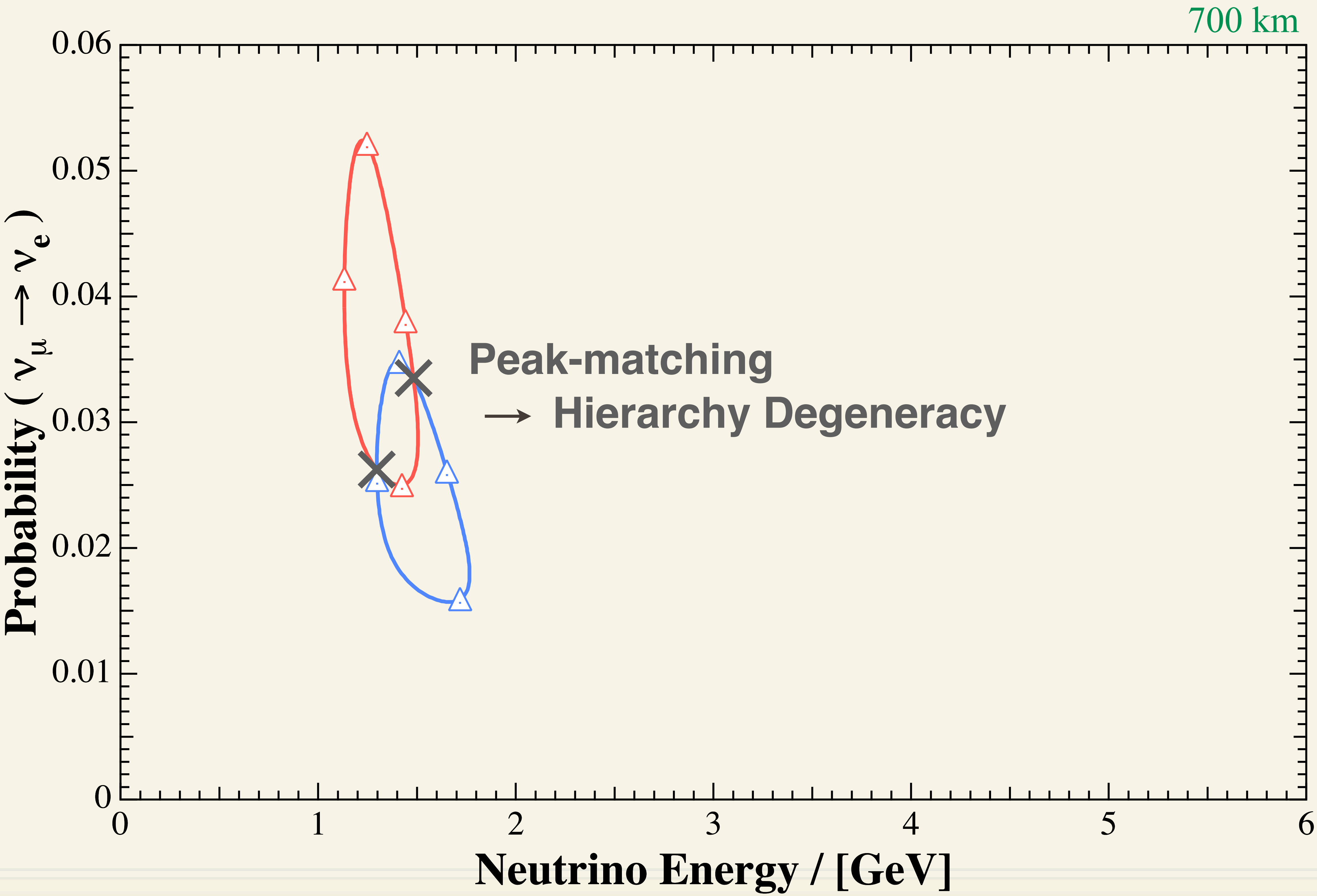
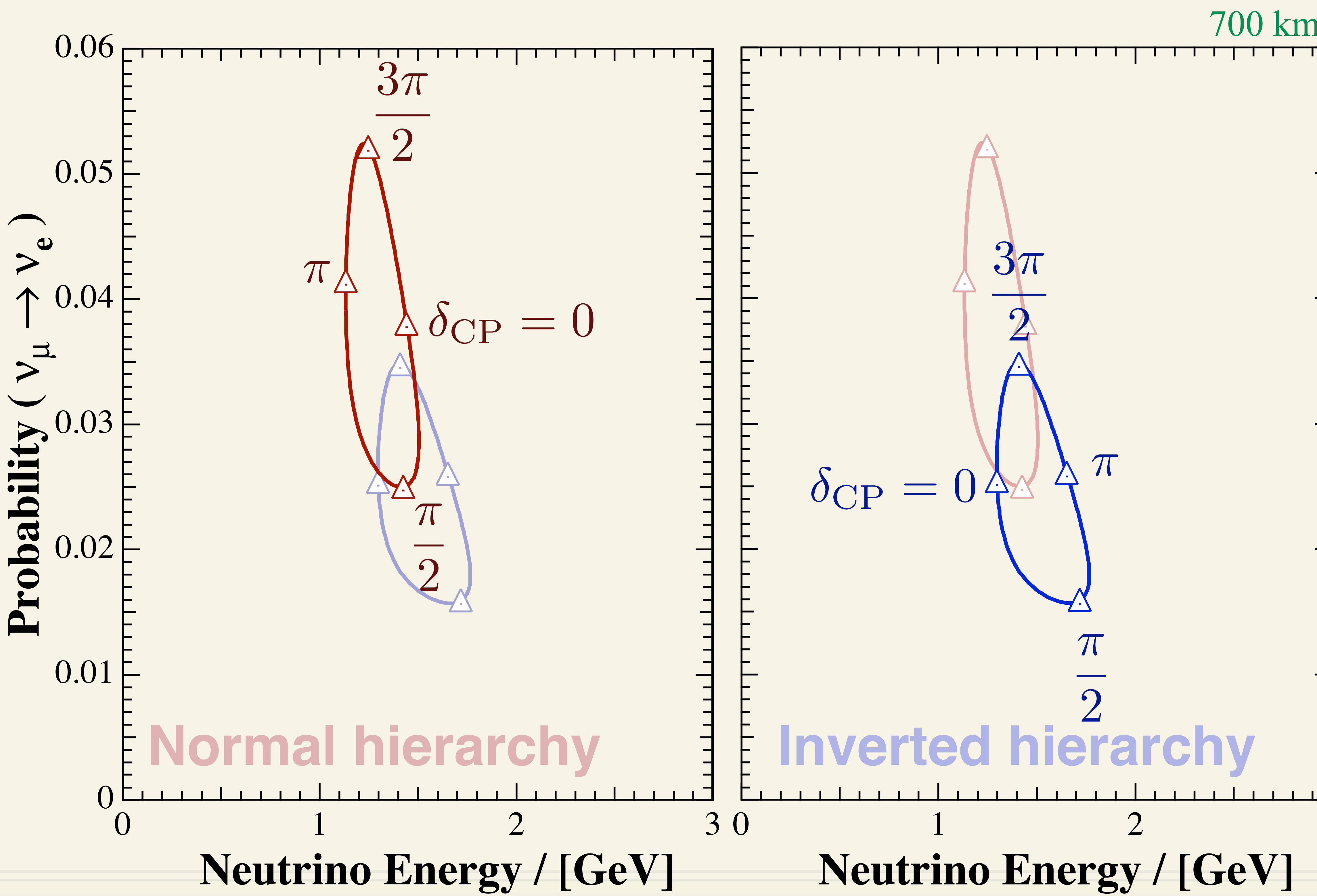


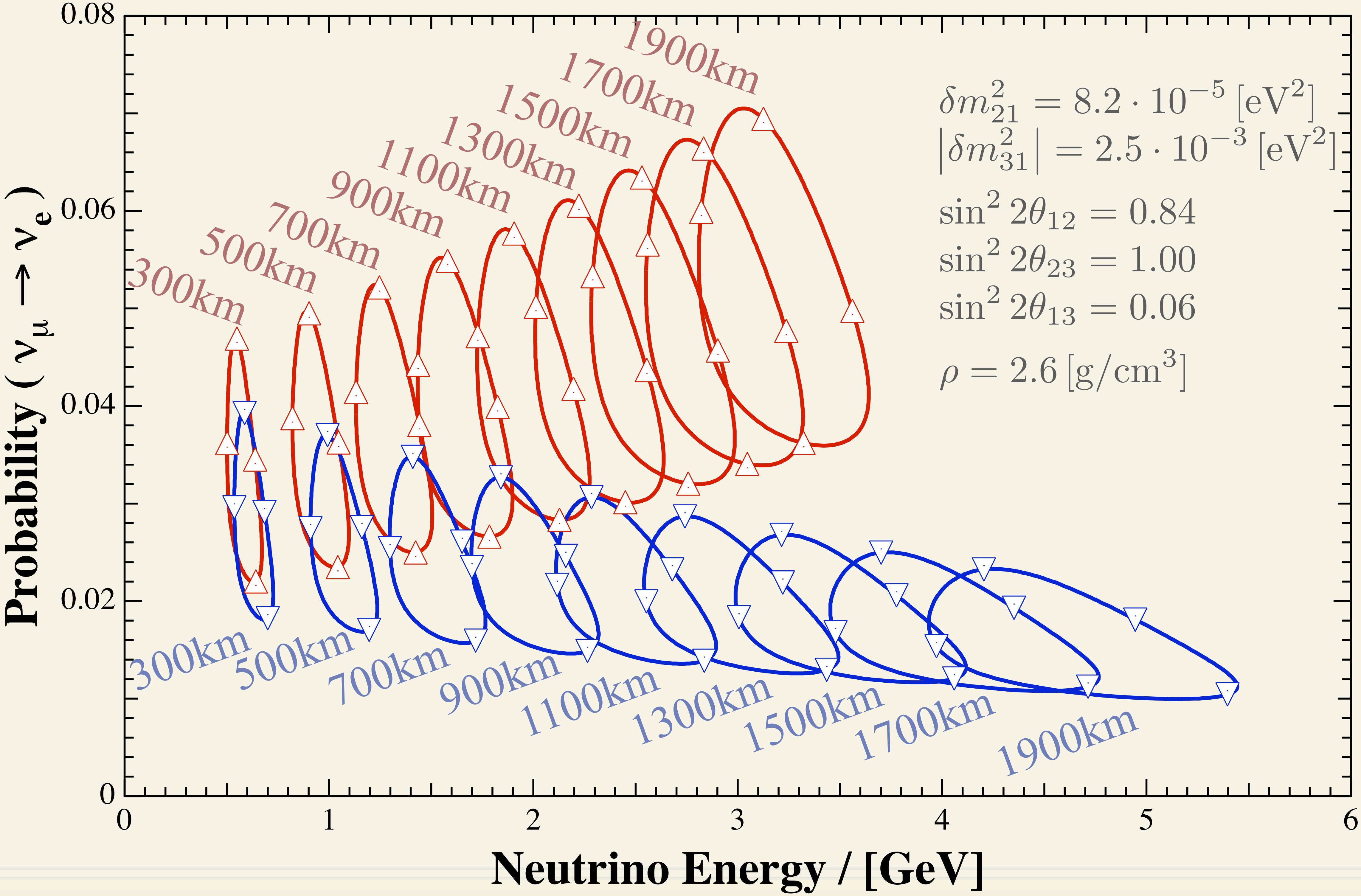
# Intersections of Loops



# Movement on the Loops



# Loops vs. Baseline Length



# Analytic Expressions

- AKS (Arafune-Koike-Sato) approximation

J. Arafune, MK, J. Sato (1997)

$$\mathrm{i}\frac{\mathrm{d}\nu}{\mathrm{d}t} = H\nu = (H_0 + H_1)\nu$$

$$\nu(x) = S(x)\nu(0)$$

$$S(x) = \mathrm{e}^{-\mathrm{i}H_0 x}\,\mathrm{Texp}\left[-\mathrm{i}\int_0^x \mathrm{d}s\, H_1^{(\mathrm{I})}(s)\right] \quad \left(H_1^{(\mathrm{I})}(x) \equiv \mathrm{e}^{\mathrm{i}H_0 x} H_1 \mathrm{e}^{-\mathrm{i}H_0 x}\right)$$

$$= \mathrm{e}^{-\mathrm{i}H_0 x} - \mathrm{e}^{-\mathrm{i}H_0 x}\,\mathrm{i}\int_0^x \mathrm{d}s\, H_1^{(\mathrm{I})}(s) + \cdots$$

$$P(\nu_\alpha \to \nu_\beta) = \left|S_{\beta\alpha}\right|^2 \quad (\{\alpha, \beta\} \in \{\mathrm{e}, \mu, \tau\})$$

- Conditions of applicability

(i) $\delta m_{21}^2 \ll \delta m_{31}^2$ ,   (ii) $a \ll \delta m_{31}^2$ ,   (iii) $\dfrac{aL}{2E} \ll 1$

$$\left(\frac{\rho}{[\mathrm{g\,cm^3}]}\frac{E}{[\mathrm{GeV}]} \ll \frac{\delta m_{31}^2}{7.56\cdot 10^{-5}\,[\mathrm{eV}^2]}\right) \quad \left(\frac{\rho}{[\mathrm{g\,cm^3}]}\frac{L}{[\mathrm{km}]} \ll 5200\right)$$

→ Short baseline approximation (typically $L \lesssim (1500 - 2000)$ [km])

# Oscillation Probability

- AKS (Arafune-Koike-Sato) second-order approximation (omitted in part)

$$P(\nu_\mu \to \nu_\mathrm{e}; E) = 4l\Big[C(E)\sin^2\Theta(E) + D(E)\Big],$$

$$C(E) = 1 + 2\frac{\Delta_\mathrm{m}}{\Delta_{31}}(1-2s_{13}^2) - \Delta_{21}\frac{j}{l}\sin\delta - \Delta_{21}\frac{\Delta_\mathrm{m}}{\Delta_{31}}\frac{j}{l}\left(\sin\delta + \frac{\Delta_{31}}{2}\cos\delta\right)$$
$$+ \frac{\Delta_{21}^2}{2}\left[\frac{j}{l}\cos\delta + (1-2s_{12}^2)\right]\frac{j}{l}\cos\delta + 3\frac{\Delta_\mathrm{m}^2}{\Delta_{31}^2}$$

$$\Theta(E) = \frac{\Delta_{31}}{2} - \frac{\Delta_\mathrm{m}}{2}(1-2s_{13}^2) + \frac{\Delta_{21}}{2}\left(\frac{j}{l}\cos\delta - s_{12}^2\right)$$
$$- \frac{\Delta_{21}}{2}\frac{\Delta_\mathrm{m}}{\Delta_{31}}\frac{j}{l}\left(\cos\delta + \frac{\Delta_{31}}{2}\sin\delta\right) + \frac{\Delta_{21}^2}{2}\left[\frac{j}{l}\cos\delta + \frac{1}{2}(1-2s_{12}^2)\right]\frac{j}{l}\sin\delta$$

$$D(E) = \frac{\Delta_{21}^2}{4}\frac{j^2}{l^2}\sin^2\delta$$

$$\Delta_{ij} \equiv \frac{\delta m_{ij}^2 L}{2E} \qquad \Delta_\mathrm{m} \equiv \frac{aL}{2E} \qquad l = c_{13}^2 s_{13}^2 s_{23}^2 \qquad j = c_{13}^2 s_{13} c_{23} s_{23} c_{12} s_{12}$$

# Peak of the Oscillation Probability

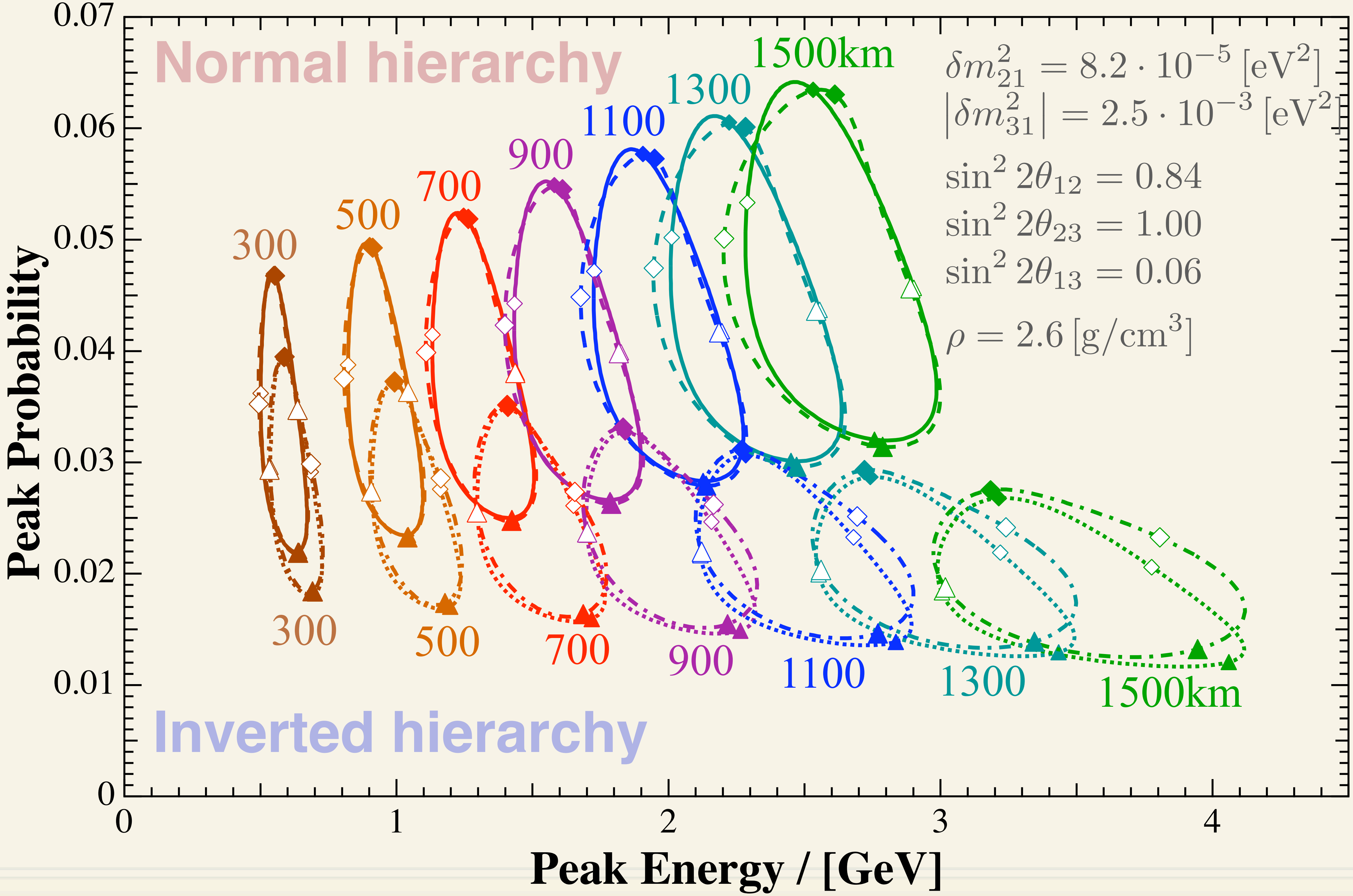
- AKS (Arafune-Koike-Sato) second-order approximation (omitted in part)

$$
\begin{aligned}
E_{\mathrm{peak},n} = \frac{|\delta m_{31}^2| L}{2\Pi} \Bigg\{ & 1 \mp \frac{\Delta_{\mathrm{m}}}{\Pi}(1-2s_{13}^2)\left(1-\frac{4}{\Pi^2}\right) \mp R s_{12}^2 \pm R\frac{j}{l}\left(\cos\delta \pm \frac{2}{\Pi}\sin\delta\right) \\
& \mp \frac{\Delta_{\mathrm{m}}}{2} R \frac{j}{l}\left[\left(1+\frac{8}{\Pi^2}-\frac{64}{\Pi^4}\right)\sin\delta \pm \frac{2}{\Pi}\left(1-\frac{8}{\Pi^2}\right)\cos\delta\right] \\
& \pm R^2 \frac{\Pi}{2}\frac{j}{l}(1-2s_{12}^2)\left(\sin\delta \mp \frac{4}{\Pi}\cos\delta\right) + \frac{\Delta_{\mathrm{m}}^2}{\Pi^2}\left(1-\frac{12}{\Pi^2}+\frac{48}{\Pi^4}\right) \\
& + R^2 \frac{j^2}{l^2}\left(\pm \Pi\cos\delta\sin\delta + 1 - 3\cos^2\delta + \frac{4}{\Pi^2}\sin^2\delta\right) \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
P_{\mathrm{peak},n} & \equiv P(\nu_\mu \to \nu_{\mathrm{e}}, E_{\mathrm{peak},n}) \\
& = 4l \Bigg\{ 1 \pm 2\frac{\Delta_{\mathrm{m}}}{\Pi}(1-2s_{13}^2) - R\Pi\frac{j}{l}\sin\delta - \frac{\Pi}{2}\Delta_{\mathrm{m}} R\frac{j}{l}\left[\left(1-\frac{4}{\Pi^2}\right)\cos\delta \mp \frac{4}{\Pi}\left(1-\frac{2}{\Pi^2}\right)\sin\delta\right] \\
& \quad + R^2\frac{\Pi^2}{2}\frac{j}{l}\left[(1-2s_{12}^2)\cos\delta \mp \frac{2}{\Pi}s_{12}^2\sin\delta\right] + \frac{\Delta_{\mathrm{m}}^2}{\Pi^2}\left(1+\frac{4}{\Pi^2}\right) \\
& \quad + \frac{1}{4}R^2\Pi^2\frac{j^2}{l^2}\left(1+\cos^2\delta \pm \frac{4}{\Pi}\cos\delta\sin\delta + \frac{4}{\Pi^2}\sin^2\delta\right) \Bigg\}
\end{aligned}
$$

$R \equiv \dfrac{\delta m_{21}^2}{|\delta m_{31}^2|}$ $\quad \Delta_{\mathrm{m}} \equiv \dfrac{aL}{2E}$ $\quad l = c_{13}^2 s_{13}^2 s_{23}^2$ $\quad j = c_{13}^2 s_{13} c_{23} s_{23} c_{12} s_{12}$ $\quad \Pi = (2n+1)\pi \ (n = 0, 1, 2, \cdots)$

# Numerical vs. Analytic



# Size of Loops

$$R \equiv \frac{\delta m^2_{21}}{|\delta m^2_{31}|}$$

$$\Delta_\mathrm{m} \equiv \frac{aL}{2E}$$

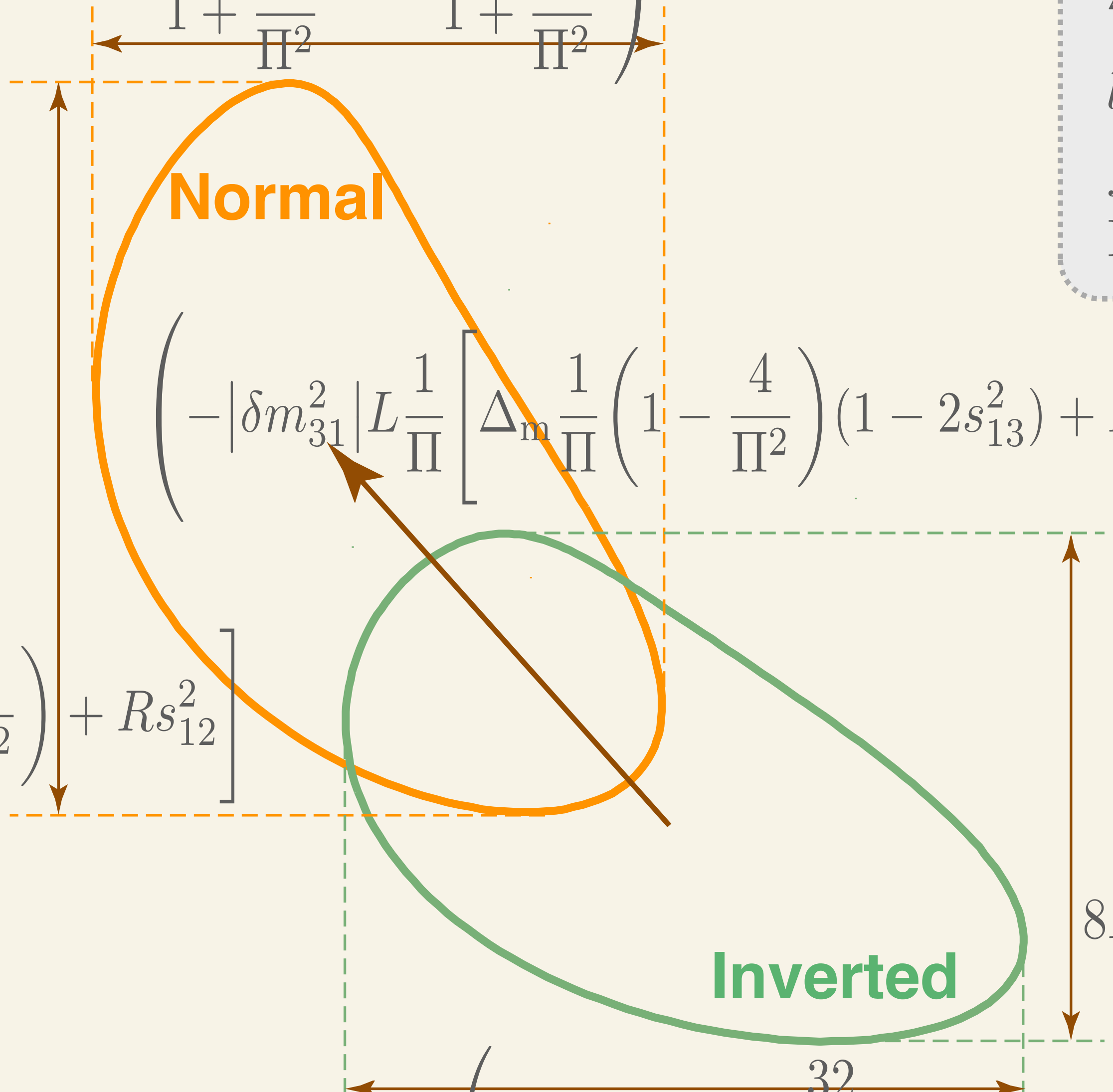
$$l = c^2_{13} s^2_{13} s^2_{23}$$

$$j = c^2_{13} s_{13} c_{23} s_{23} c_{12} s_{12}$$

$$\Pi = (2n+1)\pi \ (n = 0, 1, 2, \cdots)$$

$$\Delta_\mathrm{m} \propto L$$

$$\delta m^2_{21} L \frac{1}{\Pi} \sqrt{1+\frac{4}{\Pi^2}} \frac{j}{l} \left( 1 - \Delta_\mathrm{m} \frac{2}{\Pi} \frac{1-\frac{32}{\Pi^4}}{1+\frac{4}{\Pi^2}} - R \frac{1-2s^2_{12}}{1+\frac{4}{\Pi^2}} \right)$$




$$\left( -|\delta m^2_{31}| L \frac{1}{\Pi} \left[ \Delta_\mathrm{m} \frac{1}{\Pi} \left( 1 - \frac{4}{\Pi^2} \right) (1-2s^2_{13}) + R s^2_{12} \right], \ \Delta_\mathrm{m} \frac{16}{\Pi} l (1-2s^2_{13}) \right)$$

$$8R\Pi j \left[ 1 + \Delta_\mathrm{m} \frac{2}{\Pi} \left( 1 - \frac{2}{\Pi^2} \right) + R s^2_{12} \right]$$

$$8R\Pi j \left[ 1 - \Delta_\mathrm{m} \frac{2}{\Pi} \left( 1 - \frac{2}{\Pi^2} \right) - R s^2_{12} \right]$$

$$\delta m^2_{21} L \frac{1}{\Pi} \sqrt{1+\frac{4}{\Pi^2}} \frac{j}{l} \left( 1 + \Delta_\mathrm{m} \frac{2}{\Pi} \frac{1-\frac{32}{\Pi^4}}{1+\frac{4}{\Pi^2}} + R \frac{1-2s^2_{12}}{1+\frac{4}{\Pi^2}} \right)$$

# Baseline Length of Loop Separation

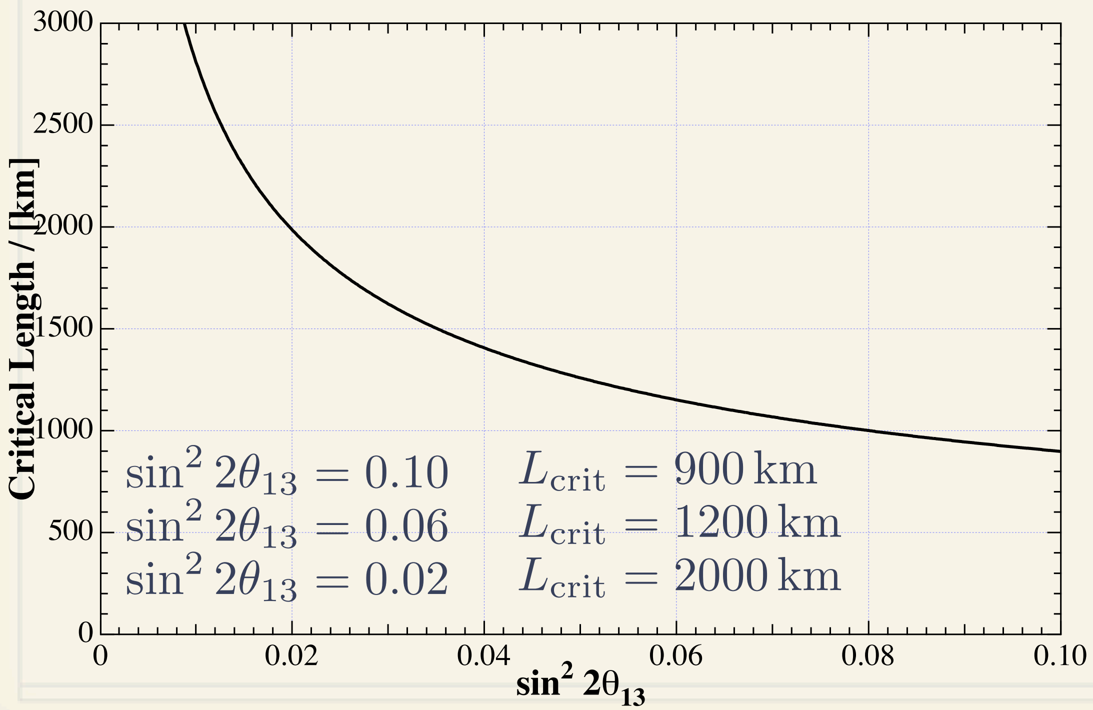
- Loops separate and the hierarchy degeneracy lifts at a long baseline $L > L_{\rm crit}$

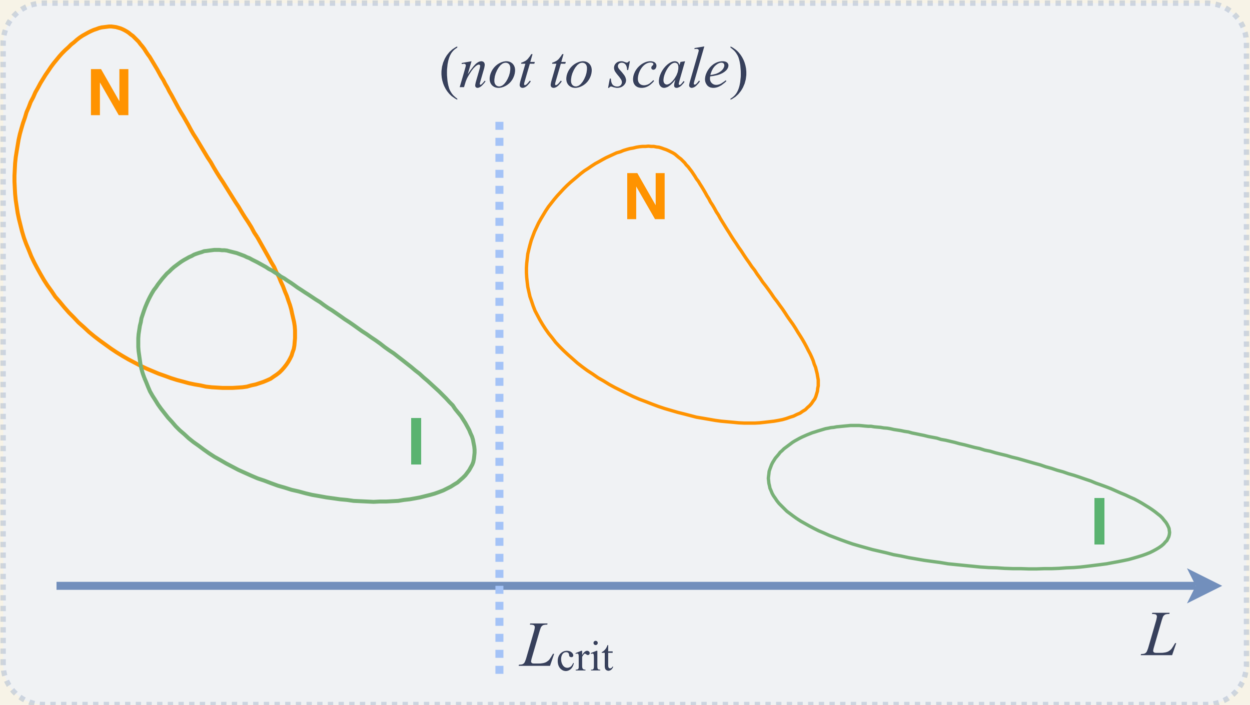
$$L_{\rm crit} = \frac{1}{a'}\frac{\delta m^2_{21}}{|\delta m^2_{31}|}\frac{1}{1-2s^2_{13}}\frac{\Pi}{1-\frac{12}{\Pi^2}+\frac{64}{\Pi^4}}\left[-\left(1-\frac{8}{\Pi^2}\right)s^2_{12}+\sqrt{\left(1-\frac{12}{\Pi^2}+\frac{64}{\Pi^4}\right)\frac{c^2_{23}c^2_{12}s^2_{12}}{s^2_{23}s^2_{13}}-\frac{4}{\Pi^2}s^4_{12}}\right]$$

*(up to first order)*

$$\simeq \frac{1}{a'}\frac{\delta m^2_{21}}{|\delta m^2_{31}|}\frac{\Pi}{\sqrt{1-\frac{12}{\Pi^2}+\frac{64}{\Pi^4}}}\frac{c_{23}c_{12}s_{12}}{s_{23}s_{13}(1-2s^2_{13})} \sim \frac{1}{s_{13}} \qquad \left(\frac{1}{a'} \equiv \frac{1}{\sqrt{2}G_{\rm F}n_{\rm e}} = \frac{5.17\cdot 10^3\,[\rm km]}{\frac{\rho}{[\rm g\,cm^{-3}]}}\right)$$

For our example parameter set,

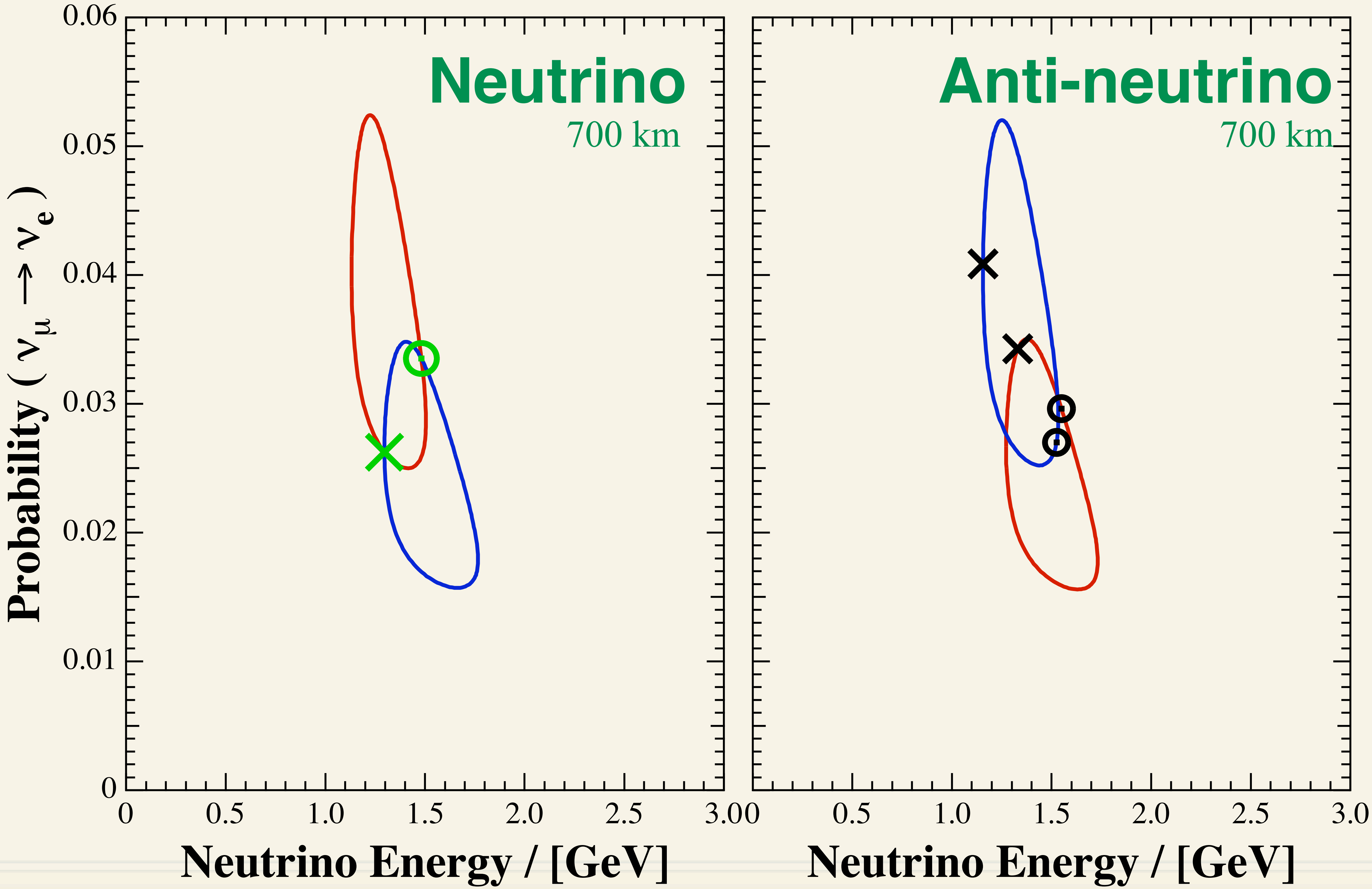


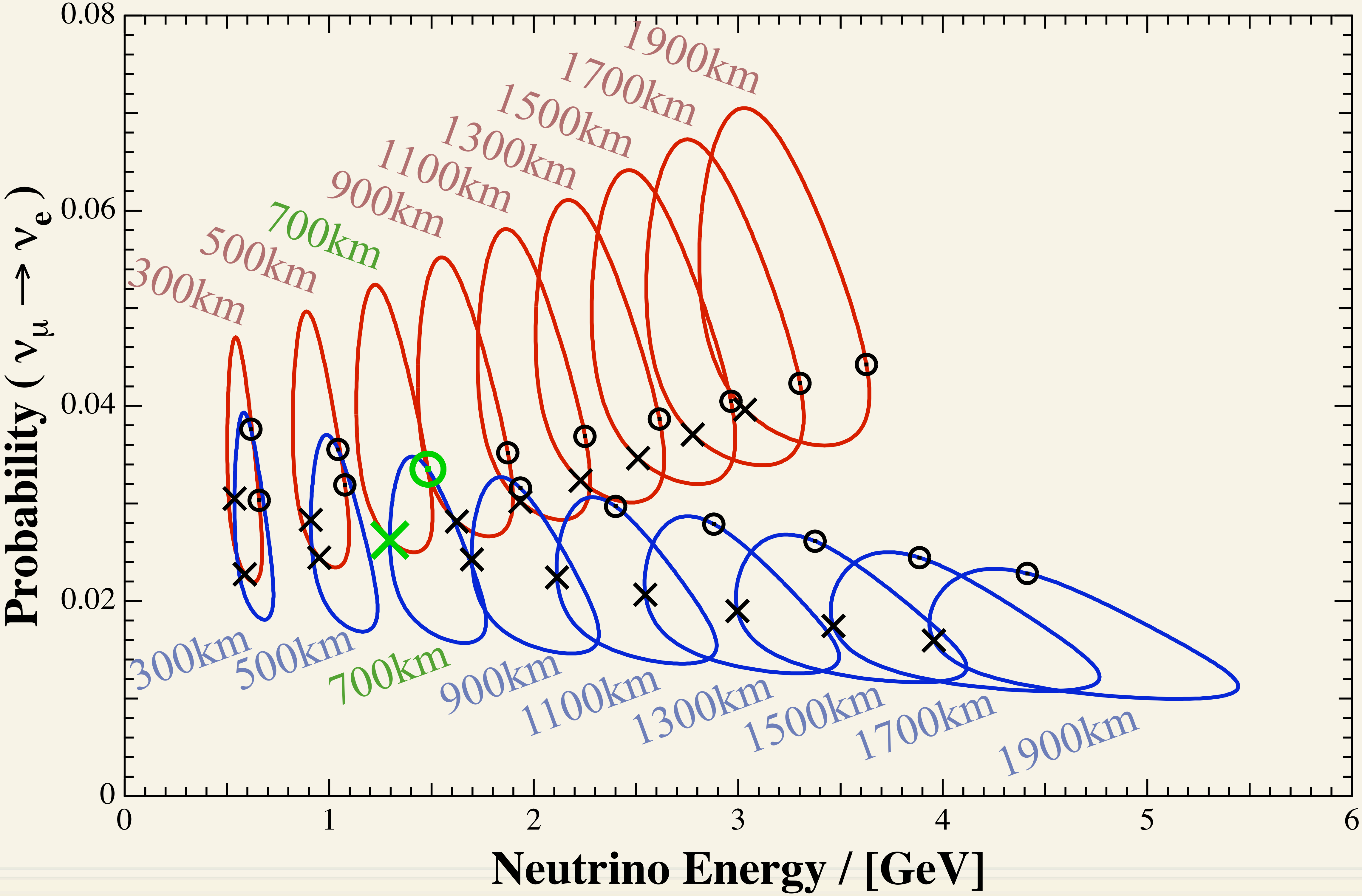
# Resolving the Degeneracy

- The way out from the hierarchy degeneracy?
  - Go for a long-length baseline, $L > L_{\rm crit}$.
  - Employ anti-neutrino beams together with neutrino beams.
  - Combine two different baseline lengths.
  - Push the detection to the lower-energy neutrinos.

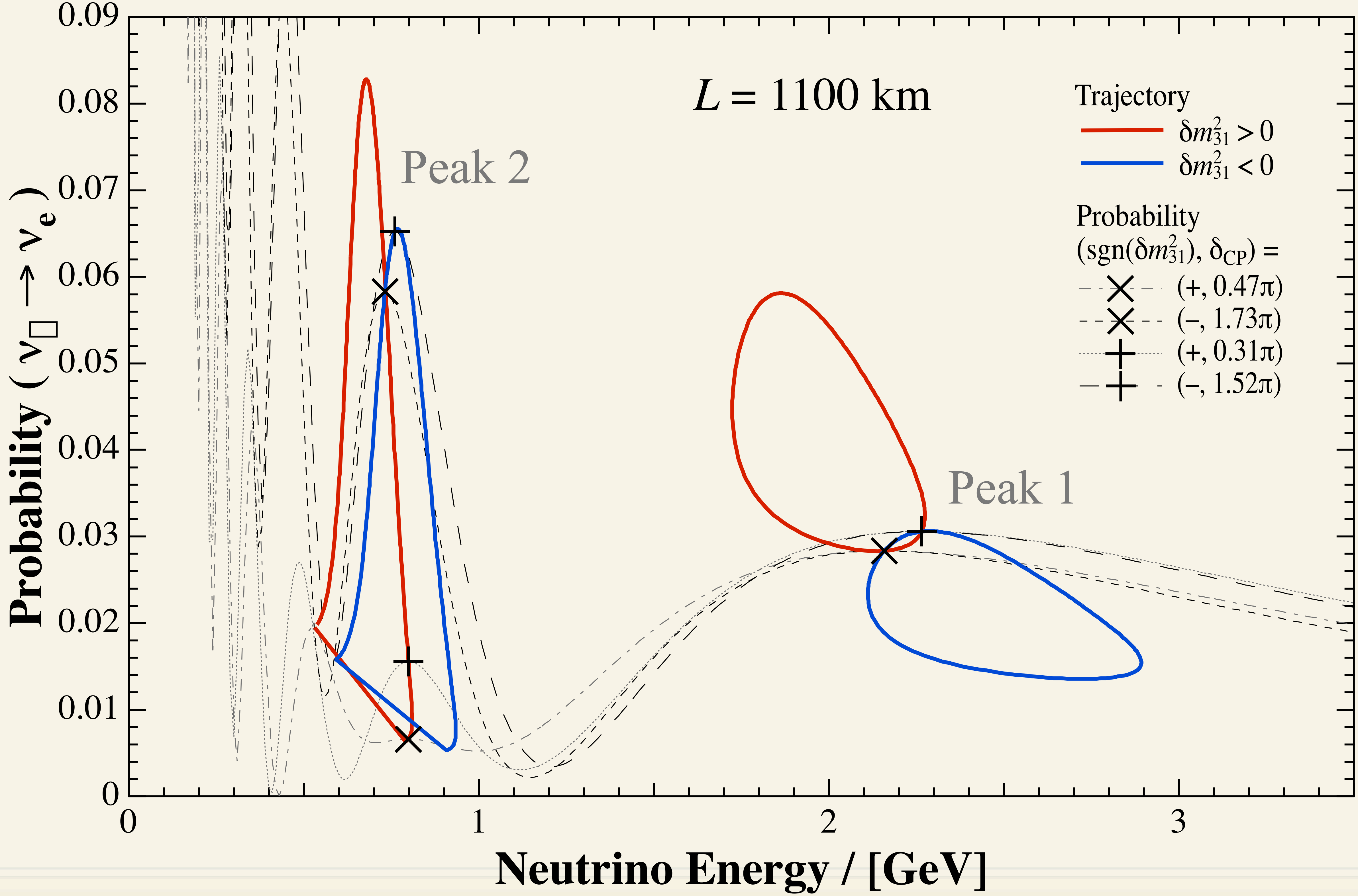
# Use of Anti-neutrinos



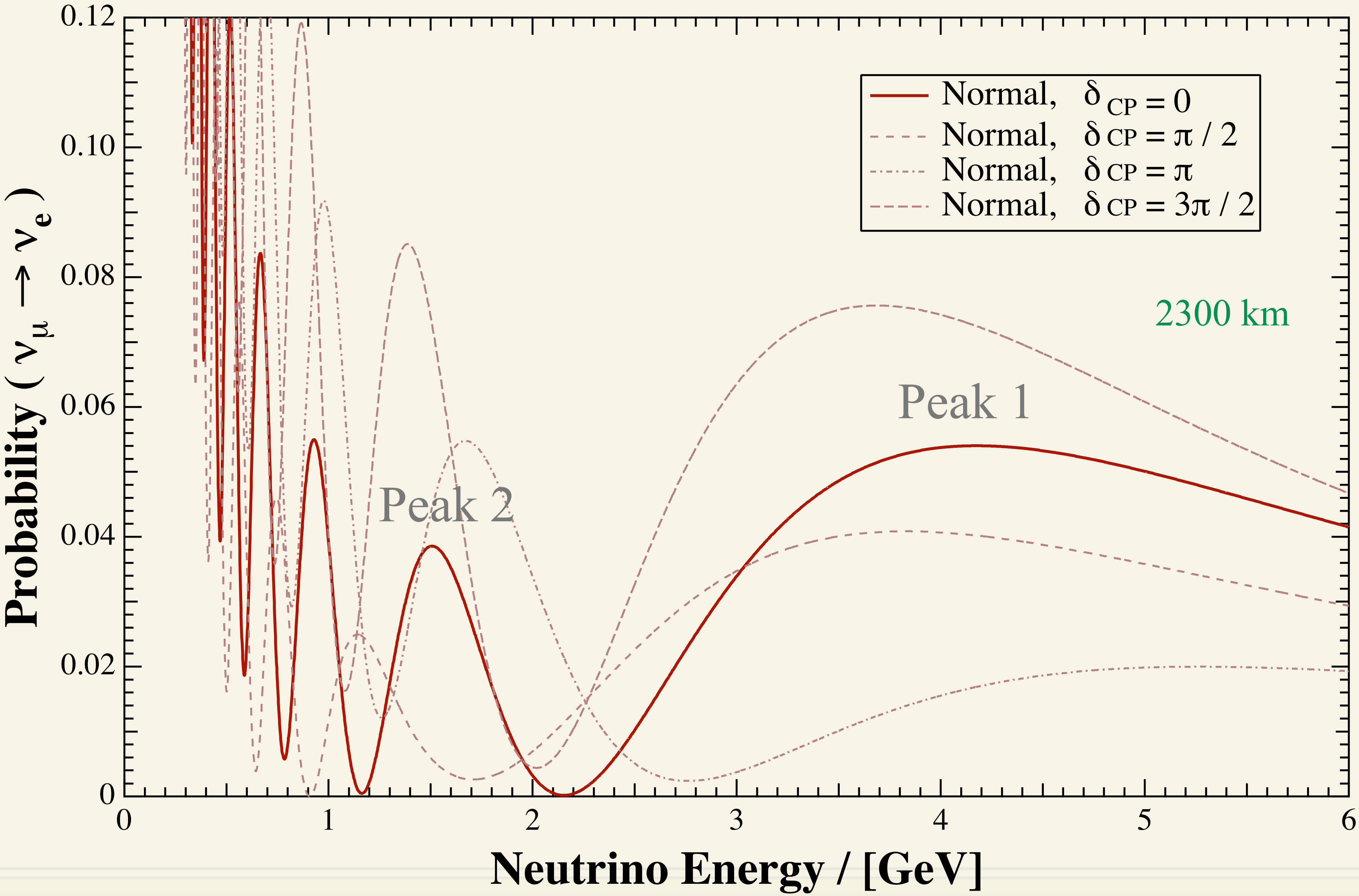
# Use of Another Distance



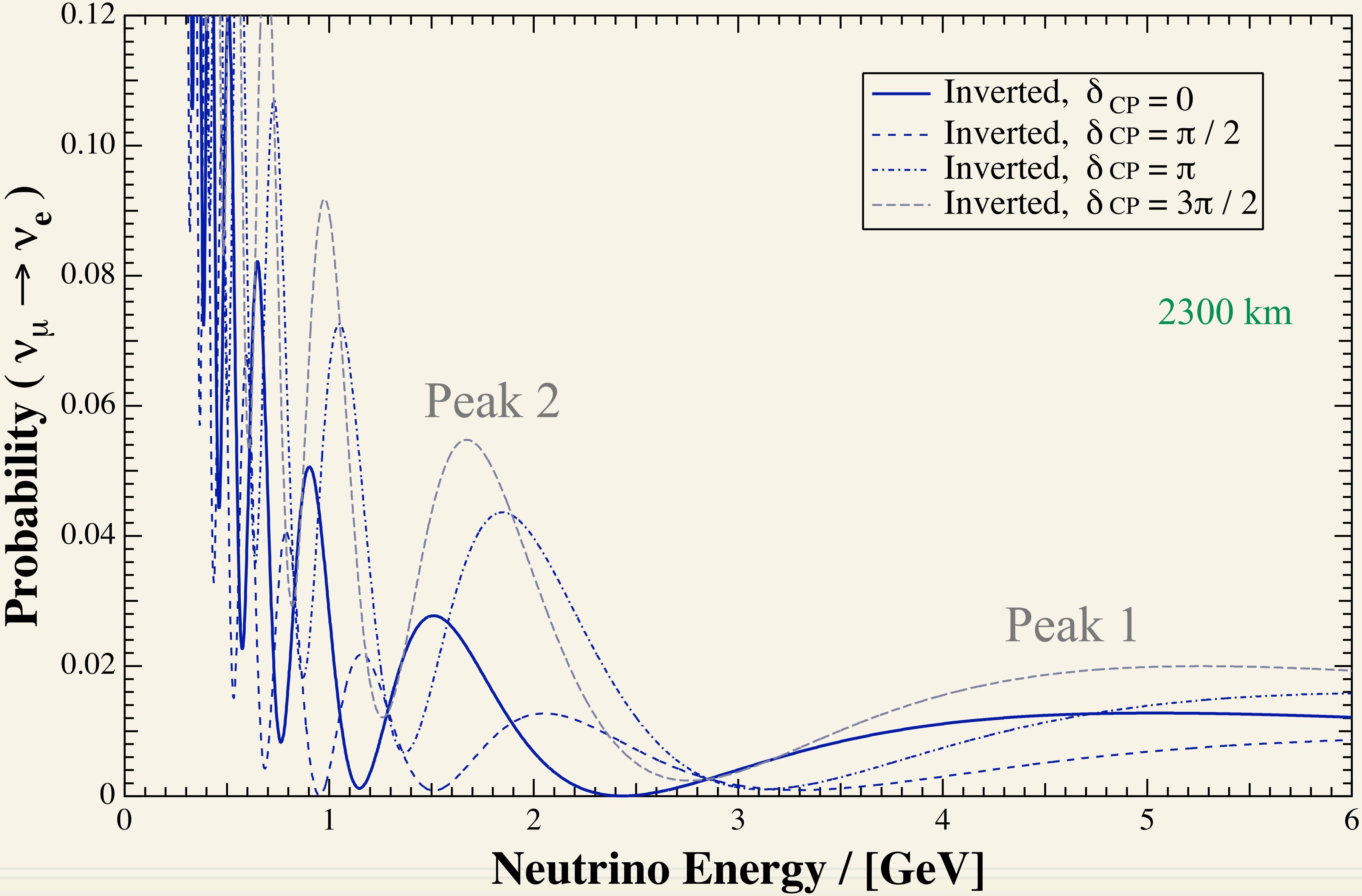
# Use of Another Peak



# Use of Another Peak



# Use of Another Peak



# Conclusions, Outlooks

- Hierarchy degeneracy may complicate the determination of the parameters.
- The peak of the oscillation is a good representative of the whole spectra.
  - Peak-matching leads to the mutually "*similar* " oscillation spectra.
- The analysis of the peak position provides a perspective of the presence and absence of parameter degeneracies in the long baseline experiments.
- The parameter-searching power can be systematically analyzed.
  - Disappearance baseline length of hierarchy degeneracy
  - Ambiguities of the oscillation parameters (*to be done*)
  - Combination of neutrinos and anti-neutrinos (*to be done*)