Anomaly of Discrete Family Symmetries and Gauge Coupling Unification

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Motivation

Recently, a number of models with a non-abelian discrete family symmetry are proposed. $(ex).\ A_4\ S_4\ D_N\ Q_N$

However it is difficult to construct a realistic (renormalizable) mlodel with only the SM Higgs or the MSSM Higgs.

+ several pairs of the Higgs

An extension of the Higgs sector may spoil the successful gauge coupling unification of the MSSM !!

$$g_3 = g_2 = g_1 = g_{gut}$$

But in string theory …

$$k_3 g_3^2 = k_2 g_2^2 = k_1 g_1^2 = g_{st}^2$$

k_i : Kac – Moody levels

GS mechanism

$$\frac{A_3}{k_3} = \frac{A_2}{k_2} = \frac{A_1}{k_1} = \frac{A_{Gravity}}{12} = \delta_{GS}$$

Motivation

- If we can define anomaly for discrete symmetries…
- If its anomaly should be canceled by the GS mechanism…

We can restrict the Kac-Moody levels

If we can arhieve the gauge coupling unification in a range of above limit, it is very interesting !!

・Plan of my talk

1. define anomaly for discrete symmetries.
2. apply the GS mechanism to discrete symmetries.
3. investigate anomaly of a non-abelian family symmety and their cancellation mechanism.
4. examine the running of gauge couplings.

Anomaly of Discrete Symmetries ?

Anomaly = a violation of a classical symmetry

at the quantum level

Non-conservation of ***Noether current***

tree level $\longrightarrow$ $\partial^\mu j_\mu = 0$

quantum level $\longrightarrow$ $\partial^\mu j_\mu \neq 0$

But we cannot define ***Noether current***

for discrete symmetries

What is anomaly of discrete symmetries ?

Fujikawa's Method

Anomalies can been defined as the Jacobian of the path-Integral measure [Phys.Lett.B260,291(1991): L.E.Ibanez , G.G.Ross]

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi \to J^{-1}\mathcal{D}\bar{\psi}\mathcal{D}\psi \qquad (\psi \to \mathrm{e}^{i\alpha(x)\gamma_5}\psi)$$

$$J^{-1} = \mathbf{exp}\left\{-\int d^4x \; i\frac{\alpha(x)}{16\pi^2}\mathrm{Tr}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right\}$$

$$\alpha(\mathbf{x}) \to \theta$$

$$(ex).\ \theta = \frac{2\pi}{N}$$

$$J^{-1} = \mathbf{exp}\left\{-\int d^4x \; i\frac{\theta}{16\pi^2}\mathrm{Tr}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right\}$$

We call this Jacobian

Anomaly of the discrete symmetry

Anomaly of discrete symmetries

Consider the discrete chiral phase rotation.

$$\psi \to \mathrm{e}^{i\theta\gamma_5}\psi \qquad \bar{\psi} \to \bar{\psi}\mathrm{e}^{i\theta\gamma_5}$$

To calculate the Jacobian of the path-integral measure, let us define the following eigenstates $\varphi_n(x)$ of $\not{D}$,

$$\not{D}\varphi_n(x) = \lambda_n \varphi_n \qquad \varphi_n^\dagger(x)\not{D} = \lambda_n \varphi_n^\dagger$$

with the completeness relations, in Euclidean space time.

$$\psi_i(x) = \sum_n a_n \varphi_{n,i}(x) \qquad \sum_n^\infty \varphi_n(y)\varphi_n^\dagger(x) = \delta^4(x-y)$$

$$\bar{\psi}_i(x) = \sum_n \bar{b}_n \varphi_{n,i}^\dagger(x) \qquad \int d^4x\ \varphi_n^\dagger(x)\varphi_m(x) = \delta_{nm}$$

Under the above transformation, the Jacobian of the path-integral measure can be written as

$$J^{-1} = det\left[\int dx^4\ \varphi_{n,i}^\dagger(x)\ \left(\mathrm{e}^{i\theta\gamma_5}\right)_{ij}\ \varphi_{m,j}(x)\right]^{-2} \equiv det\left[C_{nm}\right]^{-2}.$$

Anomaly of discrete symmetries

C_{nm} is defined as the expansion

$$\begin{aligned} C_{nm} \;&=\; \delta_{nm} + \int d^4x\ \varphi^\dagger_{n,i}(x)\,[i\theta\gamma_5]_{ij}\,\varphi_{m,j}(x) \\ &\quad + \int d^4x\ \varphi^\dagger_{n,i}(x)\frac{1}{2!}\,[i\theta\gamma_5]^2_{ij}\,\varphi_{m,j}(x) \\ &\quad + \int d^4x\ \varphi^\dagger_{n,i}(x)\frac{1}{3!}\,[i\theta\gamma_5]^3_{ij}\,\varphi_{m,j}(x) + \cdots. \end{aligned}$$

By using the completeness relations, we can derive

$$\begin{aligned} &\int d^4x\ \varphi^\dagger_{n,i}(x)\,[i\theta\gamma_5]^2_{ij}\,\varphi_{m,j}(x) \\ &\quad = \int d^4x \int d^4y\ \varphi^\dagger_{n,i}(x)\,[i\theta\gamma_5]_{ij}\,\delta_{jk}\delta^4(x-y)\,[i\theta\gamma_5]_{kl}\,\varphi_{m,l}(y) \\ &\quad = \int d^4x \int d^4y\ \varphi^\dagger_{n,i}(x)\,[i\theta\gamma_5]_{ij}\,\varphi_{p,j}(x)\varphi^\dagger_{p,k}(y)\,[i\theta\gamma_5]_{kl}\,\varphi_{m,l}(y) \\ &\quad = \tilde{C}_{np}\tilde{C}_{pm} = \tilde{C}^2_{nm} \end{aligned}$$

where

$$\tilde{C}_{nm} = \int d^4x\ \varphi^\dagger_{n,i}(x)\,[i\theta\gamma_5]_{ij}\,\varphi_{m,j}(x).$$

Anomaly of discrete symmetries

In a similar manner we can prove the identity

$$\int d^4x\ \varphi^\dagger_{n,i}(x)\,[i\theta\gamma_5]^N_{ij}\,\varphi_{m,j}(x) = \tilde{C}^N_{nm}.$$

Therefore

$$C_{nm} = \delta_{nm} + \tilde{C}_{nm} + \frac{1}{2!}\tilde{C}^2_{nm} + \frac{1}{3!}\tilde{C}^3_{nm} + \cdots,$$

and we can rewrite the Jacobian as

$$\begin{aligned} J^{-1} &= \mathbf{det}\left\{\int d^4x\ \varphi^\dagger_{n,i}(x)\left(\mathrm{e}^{i\theta\gamma_5}\right)_{ij}\varphi_{m,j}(x)\right\}^{-2} \\ &= \mathbf{exp}\left\{-2\sum_n^\infty \int d^4x\ \varphi^\dagger_{n,i}(x)\,[i\theta\gamma_5]_{ij}\,\varphi_{n,j}(x)\right\}. \end{aligned}$$

The remaining calculation is same as the conventional one.

Finally, the Jacobian form

$$J^{-1} = \mathbf{exp}\left\{-i\int d^4x\ \frac{\theta}{16\pi^2}\mathbf{Tr}\left[\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right]\right\}.$$

Anomaly of Discrete Flavor Symmetries

Consider a non-Abelian discrete family symmetry.

$$\psi_{L,\alpha} \to U_{\alpha\beta}\psi_{L,\beta} \equiv (\mathrm{e}^{iX})_{\alpha\beta}\psi_{L,\beta}$$

$$\psi_{R,\alpha} \to V_{\alpha\beta}\psi_{R,\beta} \equiv (\mathrm{e}^{iY})_{\alpha\beta}\psi_{R,\beta} \qquad \alpha,\beta = generations$$

$$UU^\dagger = VV^\dagger = 1$$

$$det(U) = \mathrm{e}^{i\eta} \quad det(V) = \mathrm{e}^{i\xi}$$

Anomaly is

$$J^{-1} = exp\left\{-Tr\int dx^4 \varphi^\dagger_{n,i,\alpha}(x)[XP_L + YP_R]_{ij,\alpha\beta}\ \varphi_{m,j,\beta}(x)\right\}$$

$$= \mathbf{exp}\left\{-i \lim_{\Lambda\to\infty} \mathbf{Tr}\int \frac{d^4k}{(2\pi)^4}\int d^4x\ \mathrm{e}^{-ikx}\,[XP_L + YP_R]\,\mathrm{e}^{-(\not{D}^\dagger\not{D}/\Lambda^2)}\mathrm{e}^{ikx}\right\}$$

Take **Tr** for generation indices

$$Tr(Y) = \xi \quad det(\mathrm{e}^{iY}) = \mathrm{e}^{i\xi}$$

$$Tr(X) = \eta \quad det(\mathrm{e}^{iX}) = \mathrm{e}^{i\eta}$$

Anomaly of Discrete Flavor Symmetries

$$J^{-1} = exp\left\{\frac{\eta}{32\pi^2}\int dx^4 \; Tr[\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}]\right\}$$

$$\times exp\left\{-\frac{\xi}{32\pi^2}\int dx^4 \; Tr[\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}]\right\}$$

$$det(U) = \mathrm{e}^{i\eta}$$

$$det(V) = \mathrm{e}^{i\xi}$$

η and ξ are phases of the transformation matricies.

These phases are corresponding to the abelian parts of the non-abelian discrete family symmetry.

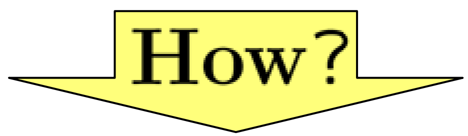
Only the abelian parts have anomaly !!

Green Schwarz(GS)-Mechanism

[Assumption]

A discrete fiamily symmetry that we observe at low energy is remnant of a more fundamental theory.

Anomaly of the discrete family symmetry should be canceled at least at the fundamental scale.



GS-Mechanism

[Phys.Lett.B149,117(1984): M.Green , J.H.Schwarz]

GS-Mechanism

String theory when compactified to 4-dimensions contains

$\mathcal{G} = SU(3)_C \times SU(2)_L \times U(1)_Y \times \underline{U(1)_{Anomalous}}$.

Consider the $U(1)_A$ transformation

$$\Phi \rightarrow \mathrm{e}^{-i\Lambda}\Phi \quad V_A \rightarrow V_A + i(\Lambda - \bar{\Lambda})$$

$\Phi, \Lambda : chiral$

$V_A : vector$

Anomaly is given by [Nuovo.Cim.A90,111(1985): K.Konishi , K.Shizuya]

$$\mathbf{Anomaly} = \underline{-i\mathcal{A}[\Lambda W^a W_a]_F}$$

$W : chiral$

This Anomaly can be canceled

$$k[SW^a W_a]_F \rightarrow k[SW^a W_a]_F \underline{+ i\mathcal{A}[\Lambda W^a W_a]_F}$$

$k : Kac - Moody\ level$

$$S \rightarrow S + i\frac{\mathcal{A}}{k}\Lambda$$

by the shift of the dilaton superfield

GS-Mechanism

Consider the Kahler potential for the S.

$$K = ln\left[S + \bar{S} - \delta_{GS} V_A - i(\bar{\Lambda} - \Lambda)\left(\frac{A}{k} - \delta_{GS}\right)\right]$$

$$V_A \rightarrow V_A + i(\Lambda - \bar{\Lambda})$$

$$S \rightarrow S + i\,\frac{A}{k}\,\Lambda$$

$$\frac{A}{k} = \delta_{GS}$$

And if for SM gauge groups

$$\frac{A_3}{k_3} = \frac{A_2}{k_2} = \frac{A_1}{k_1} = \delta_{GS} = \frac{A_{Gravity}}{12}$$

$U(1)_A$ is **Anomaly free**

Discrete Varsion of the GS-Mechanism

Consider the abelian discrete transformation.

$$\Phi \to e^{-i\theta}\Phi \qquad \theta : \text{discrete parameter}$$

$$\text{Anomaly} = -i\theta A[W^a W_a]_F$$

To cancel this Anomaly

$$k[SW^a W_a]_F \to k[SW^a W_a]_F + i\theta A[W^a W_a]_F$$

$$S \to S + i\frac{A}{k}\theta \quad \left(\eta \to \eta + \frac{A}{k}\theta\right)$$

$$S|_{\theta=\bar{\theta}=0} = \varphi + i\eta$$

But the Kahler potential is invariant

$$K = ln(S + \bar{S} - \delta_{GS}V) \to ln(S + \bar{S} - \delta_{GS}V)$$

Therefore Anomaly cancellation condition is

$$\frac{A_3 + \frac{1}{2}p}{k_3} = \frac{A_2 + \frac{1}{2}q}{k_2}$$

[Nucl.Phys.B660,332(2003):
K.S.Babu , L.Gogoladze , K.Wang]

GS-Mechanism and Unification

・**Point !!**

1, the string coupling is the v.e.v of the dilaton field

$$<\varphi>= 1/g_{ST}^2 \quad (S|_{\theta=\bar{\theta}=0} = \varphi + i\eta)$$

2, gauge couplings are defined from the g_{ST} along with the k

$$1/g_i^2 = k_i/g_{ST}^2$$

3, the condition of gauge coupling unification changes

$$k_3 g_3^2 = k_2 g_2^2 = k_1 g_1^2 = g_{ST}^2$$

Anomaly cancellation condition and gauge coupling unification are closely related.

$$k_3 g_3^2 = k_2 g_2^2 = k_1 g_1^2 = g_{ST}^2$$

$$\frac{A_3 + \frac{1}{2}p}{k_3} = \frac{A_2 + \frac{1}{2}q}{k_2}$$

(ex). Z_N

MSSM + R-parity + See-Saw

$$W = QH_dD^c + QH_uU^c + LH_dE^c + LH_u\nu_R^c + \underline{M_R\nu_R^c\mu_R^c}$$

$U(1)_B$ is conserved at the classical level, while...

$U(1)_L$ is violated because of Majorana masses.

However

Z_N subgroups are still intact.

$$U(1)_B \rightarrow (Z_N)_B \quad U(1)_L \rightarrow (Z_N)_L$$

The $(Z_N)_B$ and $(Z_N)_L$ charge assignments are

	Q	U^c	D^c	L	E^c	ν_R	H^u	H^d
$(Z_N)_L$	aN	bN	cN	$\frac{N}{2}+dN$	$\frac{N}{2}+eN$	$\frac{N}{2}+fN$	gN	hN
$(Z_M)_B$	$B+iM$	$-B+jM$	$-B+kM$	lM	mM	nM	oM	pM

(ex). Z_N

- $(Z_N)_L$

The anomaly coefficients are calculated to be

$$2\mathcal{A}_3 = \frac{N}{2}[12a + 6b + 6c],$$
$$2\mathcal{A}_2 = \frac{N}{2}[18a + 3 + 6d + 2g + 2h].$$

$$\frac{A_3}{k_3} = \frac{A_2}{k_2}$$

Therefore GS cancellation condition become

$$\frac{k_3}{k_2} = \frac{12a + 6b + 6c}{18a + 3 + 6d + 2g + 2h} = \frac{even}{odd}.$$

- $(Z_N)_B$

$$\frac{k_3}{k_2} = \frac{9B + (9i + 3\ell + o + p)M}{(6i + 3j + 3k)M} = \frac{even \text{ or } odd}{even \text{ or } odd}$$

(ex). Z_N

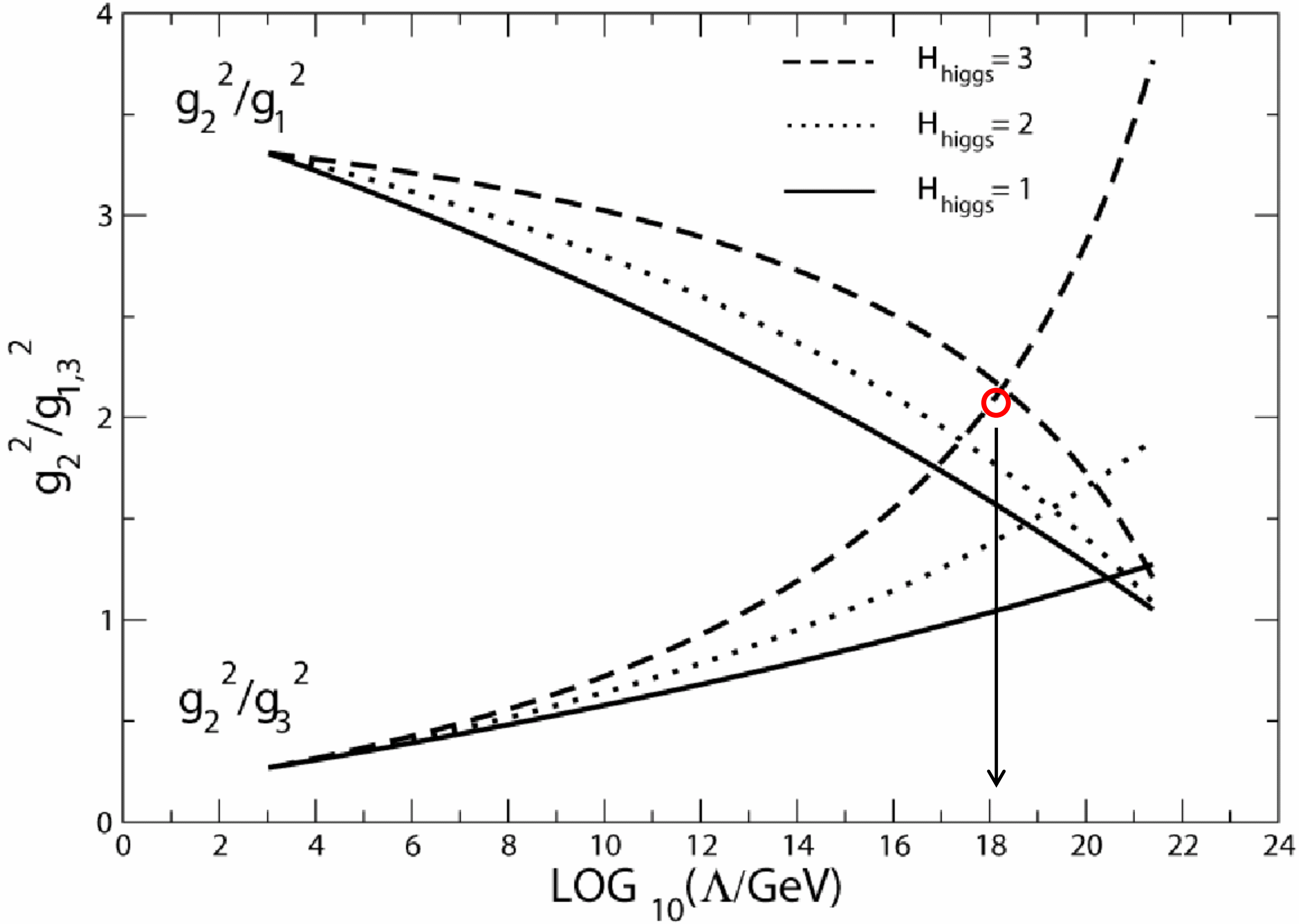
Since it is difficult to build realistic models with higher Kac-Moody levels in string theory, we adopt the lowest levels.

$$k_3 = 2\ ,\ \ k_2 = 1$$

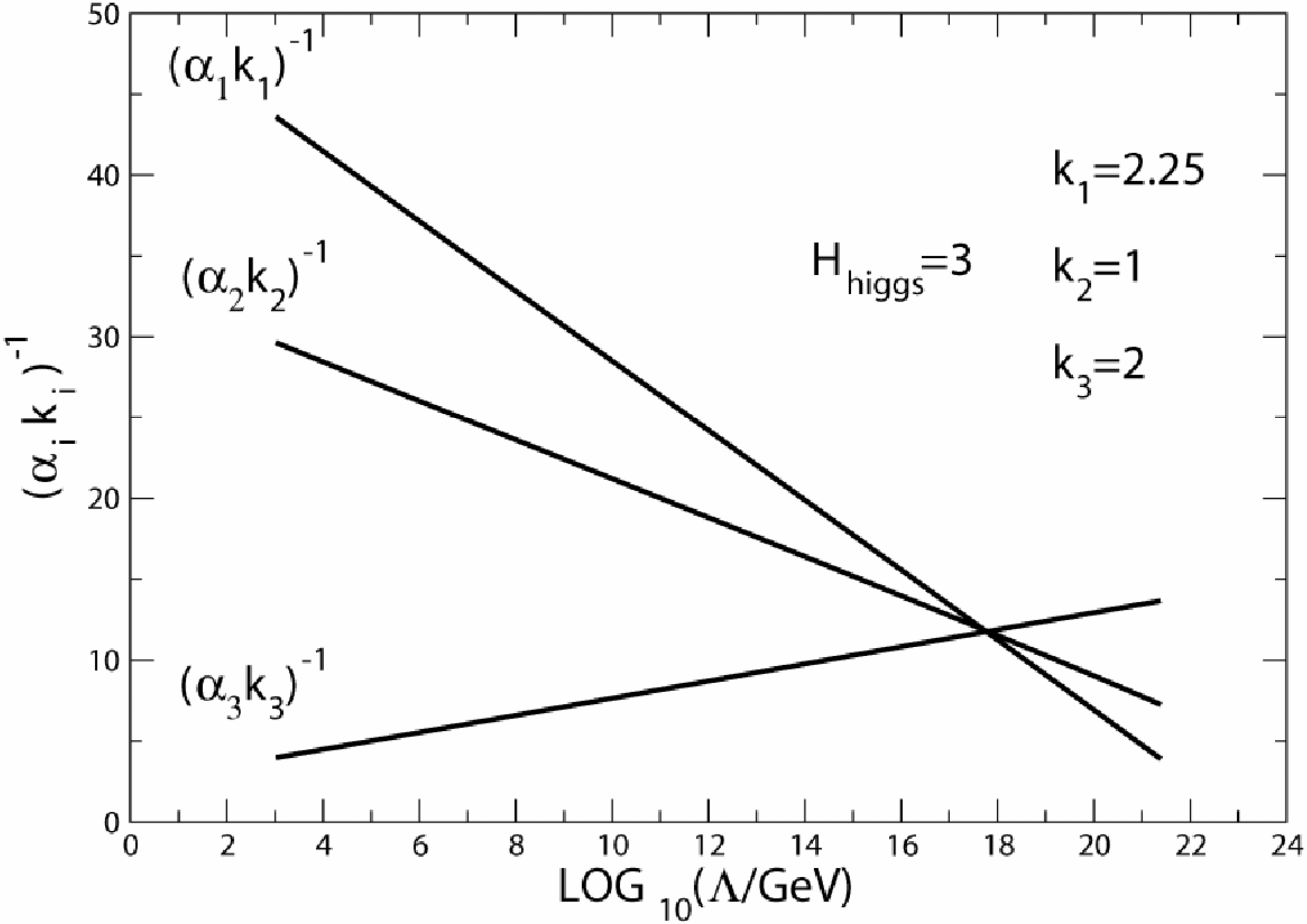
Gauge coupling unification

$$2g_3^2 = g_2^2 = k_1 g_1^2 = g_{st}^2$$

$$\frac{g_2^2}{g_3^2} = \frac{k_3}{k_2} = 2$$



g_2^2/g_1^2
g_2^2/g_3^2
$H_{higgs} = 3$
$H_{higgs} = 2$
$H_{higgs} = 1$
$g_2^2/g_{1,3}^2$
LOG $_{10}(\Lambda/\mathrm{GeV})$
0
1
2
3
4
0
2
4
6
8
10
12
14
16
18
20
22
24



$(\alpha_1 k_1)^{-1}$
$(\alpha_2 k_2)^{-1}$
$(\alpha_3 k_3)^{-1}$
$H_{higgs}=3$
$k_1=2.25$
$k_2=1$
$k_3=2$
$(\alpha_i k_i)^{-1}$
$LOG_{10}(\Lambda/GeV)$
0
2
4
6
8
10
12
14
16
18
20
22
24
0
10
20
30
40
50

Discrete Varsion of the GS-Mechanism

The anomaly cancellation condition of an abelian discrete symmetry is

$$\frac{A_3 + \frac{1}{2}p}{k_3} = \frac{A_2 + \frac{1}{2}q}{k_2}$$

But our main motivation is to investigate anomaly of a non-abelian discrete family symmetry !!

But even if we consider a non-abelian discrete family symmetry….

$$J^{-1} = exp\left\{\frac{\eta}{32\pi^2}\int dx^4\ Tr[\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}]\right\}$$

$$\times exp\left\{-\frac{\xi}{32\pi^2}\int dx^4\ Tr[\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}]\right\}$$

$$det(U) = e^{i\eta}$$

$$det(V) = e^{i\xi}$$

Only the abelian parts have anomaly !!

(ex). Q6

[Phys.Rev.D71,056006(2005): K.S.Babu , J.Kubo]

Q_6 has 12 elements

$$\mathcal{G} = \{E, A_{Q_6}, (A_{Q_6})^2, \cdots, (A_{Q_6})^5, B_Q, A_{Q_6}B_Q, (A_{Q_6})^2 B_Q, \cdots, (A_{Q_6})^5 B_Q\}$$

$$A_{Q_6} = \begin{pmatrix} cos\ \phi_6 & sin\ \phi_6 \\ -\sin\ \phi_6 & cos\ \phi_6 \end{pmatrix}_{\phi_6 = \frac{2\pi}{6}} \qquad B_Q = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

And there are 6 Irreps (2 doublets and 4 singlets).

$$doublet \ \ddagger \ \mathbf{2} \quad \mathbf{2}^{'}$$

$$singlet \ \ddagger \ \mathbf{1} \quad \mathbf{1}^{'} \quad \mathbf{1}^{''} \quad \mathbf{1}^{'''}$$

The abelian parts of Q_6 are Z_6 and Z_4 .

$$(A_{Q6})^6 = 1 \quad \rightarrow \quad Z_6$$

$$(B_{Q6})^4 = 1 \quad \rightarrow \quad Z_4$$

The assignment of Q_6 representations are

	Q, L	U^c, D^c, E^c, N^c	H^u, H^d	Q_S, L_S	$U_S^c, D_S^c, E_D^c, N_S^c$	H_S^u, H_S^d
Q_6	2	$2^{'}$	$2^{'}$	$1^{'}$	$1^{'''}$	$1^{'''}$

(ex). Q6

- Z_6 - Anomaly

$\Phi_\alpha = Q, L$

$\Psi_\alpha = U^c, D^c, E^c, N^c, H^u, H^d$

$$\Phi_\alpha \to \begin{pmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{\alpha\beta} \Phi_\beta$$

$$\Psi_\alpha \to \begin{pmatrix} -1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & e^{i\frac{2\pi}{6}3} \end{pmatrix}_{\alpha\beta} \Psi_\beta$$

$det = 1$

$det = e^{i\frac{2\pi}{6}3}$

Calculate anomaly coefficients

$$2A_3 = \underset{(Q)}{0} \cdot 2 + \underset{(D^c)}{3} + \underset{(U^c)}{3} = 6 \ (mod\ 6) = 0$$

$$2A_2 = \underset{(Q)}{0} \cdot 3 + \underset{(L)}{0} + \underset{(H_u)}{3} + \underset{(H_d)}{3} = 6 \ (mod\ 6) = 0$$

Z_6 part is anomaly free.

(ex). Q6

- Z_4 - Anomaly

$$\Phi_\alpha = Q, L$$

$$\Phi_\alpha \to \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & e^{i\frac{2\pi}{4}2} \end{pmatrix}_{\alpha\beta} \Phi_\beta$$

$$det = e^{i\frac{2\pi}{4}}\ \underline{2}$$

$$\Psi_\alpha = U^c, D^c, E^c, N^c, H^u, H^d$$

$$\Psi_\alpha \to \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & e^{i\frac{2\pi}{4}1} \end{pmatrix}_{\alpha\beta} \Psi_\beta$$

$$det = e^{i\frac{2\pi}{4}\underline{(-1)}}$$

Calculate anomaly coefficients

$$2A_3 = \underset{(Q)}{2 \cdot 2} - \underset{(D^c)}{1} - \underset{(U^c)}{1} = 2\ (mod\ 4)$$

$$2A_2 = \underset{(Q)}{2 \cdot 3} + \underset{(L)}{2} - \underset{(H_u)}{1} - \underset{(H_d)}{1} = 6\ (mod\ 4)$$

Z_4 part is anomalous.

(ex). Q6

The condition of anomaly cancellation is

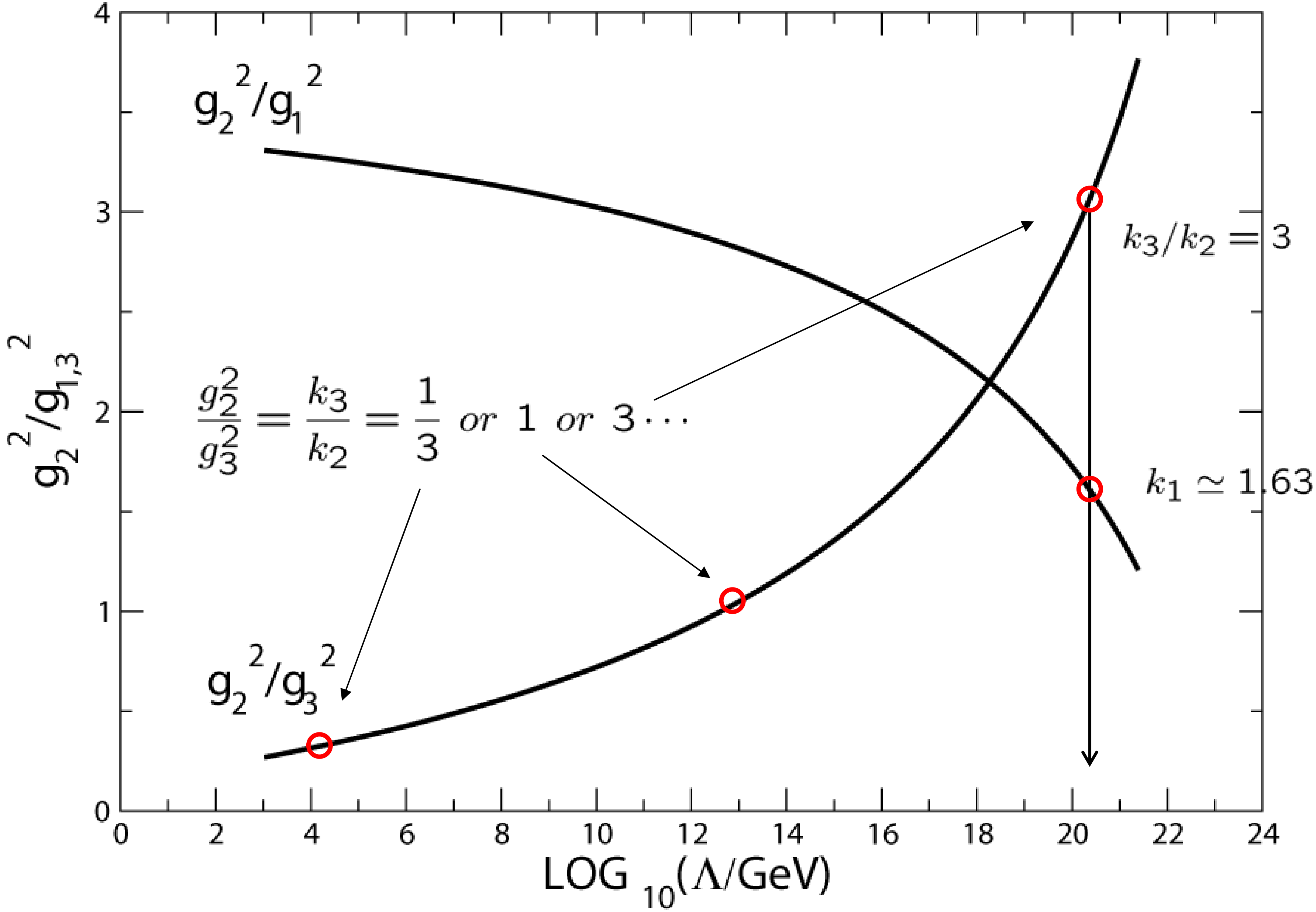
$$\frac{A_3}{k_3} = \frac{A_2}{k_2} \;\rightarrow\; \frac{1(mod\ 2)}{k_3} = \frac{1(mod\ 2)}{k_2}$$

For example, $k_3/k_2 = 1/3\ or\ 1\ or\ 3\cdots$.

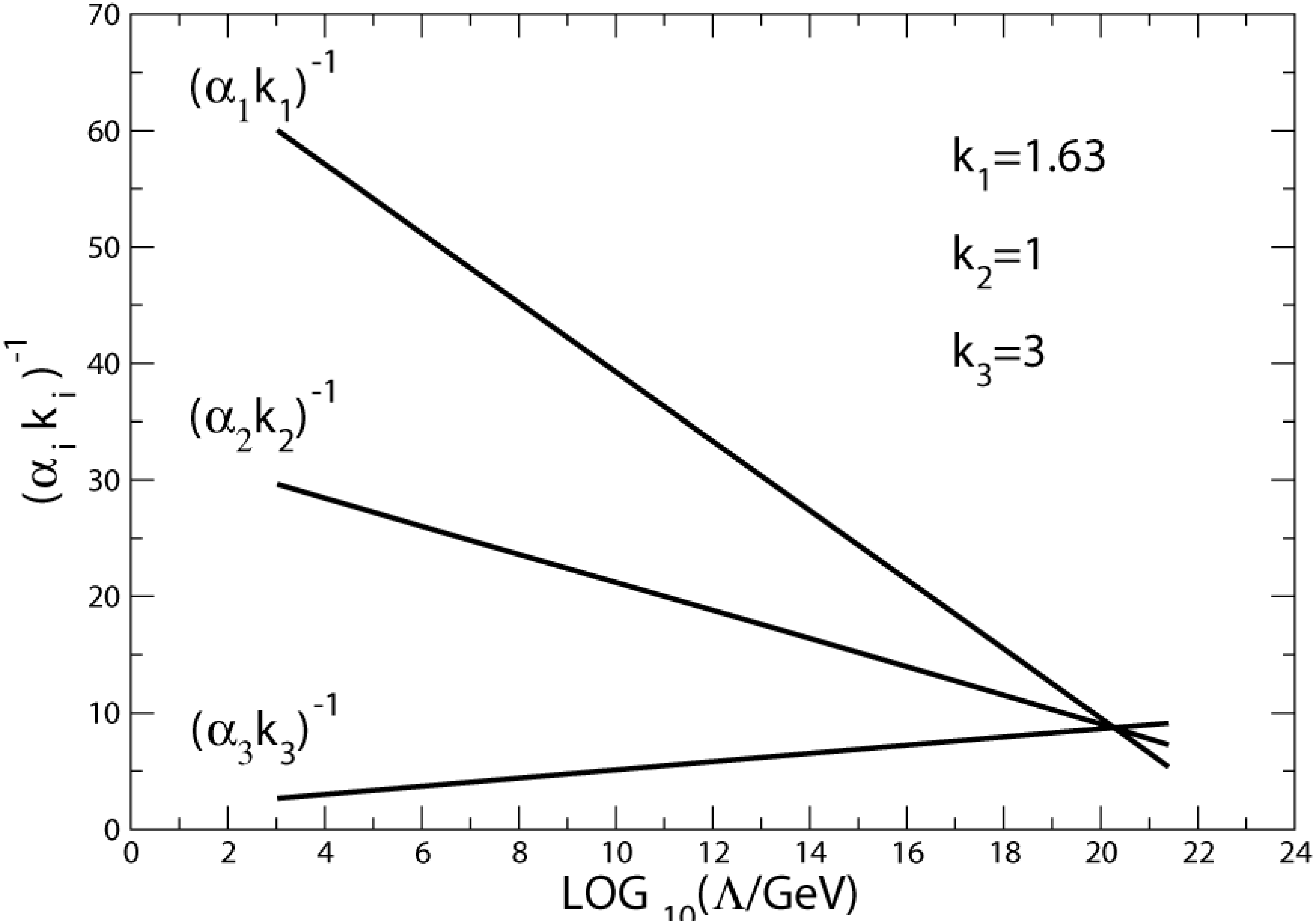
Gauge coupling unification

$$k_3 g_3^2 = k_2 g_2^2 = k_1 g_1^2 = g_{ST}^2$$

$$\frac{g_2^2}{g_3^2} = \frac{k_3}{k_2} = \frac{1}{3}\ or\ 1\ or\ 3\cdots$$



g_2^2/g_1^2
g_2^2/g_3^2
$g_2^2/g_{1,3}^2$
$\frac{g_2^2}{g_3^2} = \frac{k_3}{k_2} = \frac{1}{3}$ or 1 or 3 $\cdots$
$k_3/k_2 = 3$
$k_1 \simeq 1.63$
LOG $_{10}(\Lambda/\mathrm{GeV})$
0 2 4 6 8 10 12 14 16 18 20 22 24
0 1 2 3 4



70
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$(\alpha_1 k_1)^{-1}$
$(\alpha_2 k_2)^{-1}$
$(\alpha_3 k_3)^{-1}$
k_1=1.63
k_2=1
k_3=3
$(\alpha_i k_i)^{-1}$
LOG $_{10}$(Λ/GeV)

Summary

- We defined anomaly of discrete symmetries as the anomalous Jacobian of the path-integral measure.
- By a similar way, anomaly of non-abelian discrete family symmetries can be defined, too.
- But only the abelian parts contribute to anomaly.
- The anomaly can be canceled by the discrete version of the GS mechanism.
- If we assume that the anomaly should be canceled by the GS mechanism, the GS cancellation condition have non-trivial influence on the gauge coupling unification.

fin

Pontryagin Index

Consider the Jacobian of Z_N

$$J^{-1} = \exp\left\{-\int d^4x \ i\frac{\alpha}{32\pi^2}\mathbf{Tr}\left[\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right]\right\}.$$

In Euclid space time,

$$\int dx^4 \ \frac{1}{32\pi^2}\mathbf{Tr}\left[\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\right] = \nu$$

Is called the Pontryagin index, which is an integer.

Then, the Jacobian of Z_N can be written as

$$J^{-1} = \mathrm{e}^{\frac{2\pi i}{N}\nu Q} \qquad Q = \sum_i q_i$$

Therefore if $Q = aN(a = 0, 1, 2...)$,

$$J^{-1} = \mathrm{e}^{2\pi i a\nu} = 1$$

There are no Anomaly.

Q6

Q6 has 12 elements

$$\mathcal{G} = \{E, A_{Q_6}, (A_{Q_6})^2, \cdots, (A_{Q_6})^5, B_Q, A_{Q_6}B_Q, (A_{Q_6})^2 B_Q, \cdots, (A_{Q_6})^5 B_Q\}$$

$$A_{Q_6} = \begin{pmatrix} cos\ \phi_6 & sin\ \phi_6 \\ -\sin\ \phi_6 & cos\ \phi_6 \end{pmatrix}_{\phi_6 = \frac{2\pi}{6}} \qquad B_Q = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

and 6 Irreps (2 doublets and 4 singlets).

$$doublet \ \ddagger \ \mathbf{2} \quad \mathbf{2}'$$

$$singlet \ \ddagger \ \mathbf{1} \quad \mathbf{1}' \quad \mathbf{1}'' \quad \mathbf{1}'''$$

Direct production rules are

$$\mathbf{1}' \times \mathbf{1}' = \mathbf{1} \qquad \mathbf{1}'' \times \mathbf{1}'' = \mathbf{1}' \qquad \mathbf{1}''' \times \mathbf{1}''' = \mathbf{1}'$$

$$\mathbf{1}' \times \mathbf{1}''' = \mathbf{1}'' \quad \mathbf{1}' \times \mathbf{1}'' = \mathbf{1}''' \quad \mathbf{1}'' \times \mathbf{1}''' = \mathbf{1}$$

$$\mathbf{2} \times \mathbf{2}' = \mathbf{1}'' + \mathbf{1}''' + \mathbf{2} \qquad \mathbf{2} \times \mathbf{1}' = \mathbf{2} \quad \mathbf{2} \times \mathbf{1}'' = \mathbf{2}'$$

$$\mathbf{2}' \times \mathbf{2}' = \mathbf{1} + \mathbf{1}' + \mathbf{2}' \qquad \mathbf{2} \times \mathbf{1}''' = \mathbf{2}' \quad \mathbf{2}' \times \mathbf{1}' = \mathbf{2}'$$

$$\mathbf{2} \times \mathbf{2} = \mathbf{1} + \mathbf{1}' + \mathbf{2}' \qquad \mathbf{2}' \times \mathbf{1}'' = \mathbf{2} \quad \mathbf{2}' \times \mathbf{1}''' = \mathbf{2}$$

Q6

Transformation properties of each Irreps are

$$
\begin{array}{lll}
 & A_{Q_6} & B_Q \\
\mathbf{2} & \to \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}\mathbf{2} & \to \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\mathbf{2} \\
\mathbf{2}' & \to \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}\mathbf{2}' & \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{2}' \\
\mathbf{1} & \to \mathbf{1} & \to \mathbf{1} \\
\mathbf{1}' & \to \mathbf{1}' & \to \mathrm{e}^{i\frac{2\pi}{4}2}\ \mathbf{1} \\
\mathbf{1}'' & \to \mathrm{e}^{i\frac{2\pi}{6}3}\ \mathbf{1}'' & \to \mathrm{e}^{i\frac{2\pi}{4}3}\ \mathbf{1}'' \\
\mathbf{1}''' & \to \mathrm{e}^{i\frac{2\pi}{6}3}\ \mathbf{1}''' & \to \mathrm{e}^{i\frac{2\pi}{4}1}\ \mathbf{1}'''
\end{array}
$$